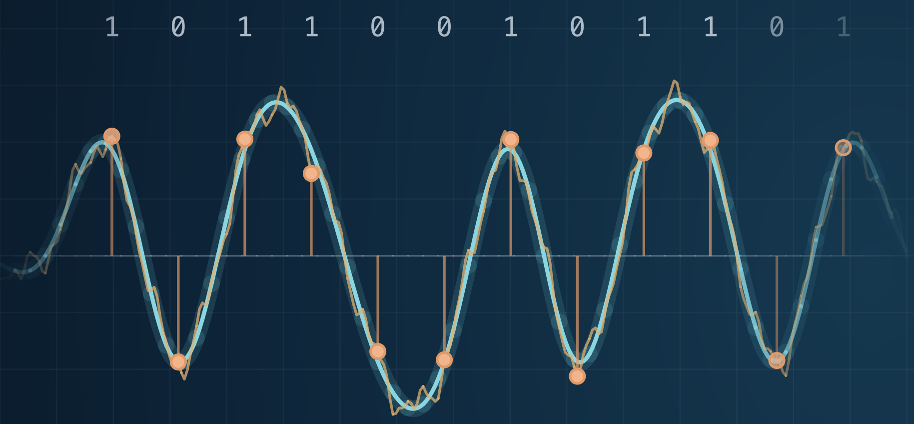
SAMPLING · DETECTION · MODULATION · CODING

Digital Communications

Formula and Notation Reference



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CONVENTIONS · FORMULAS · NOTATION

Contents

The conventions used throughout the course, every formula it establishes, and every symbol it defines. Nothing here is derived; the derivations are in the lecture notes.

How to read it

Part 1 states the conventions, the two expressions every error probability comes from, and the transform pair Chapters 1 and 2 use. Part 2 is the summary of formulas, in the order the course establishes them. Part 3 defines every symbol. Nothing here is derived: where a result needs an argument, the argument is in the lecture notes chapter named beside it.

PART

1

Conventions

These hold everywhere in the course, without local variation. Half the factor-of-two errors in this subject come from mixing two of them in one line, so where a result depends on one it is restated at that point rather than assumed.

CONVENTION	STATEMENT
Energy and power	Normalised: the resistance is taken as $1\ \Omega$, so instantaneous power is $\lvert x(t)\rvert^2$ and energy is its integral. Signal energy is E, energy per bit E_b, energy per symbol E_s, and $E_s = (\log_2 M)E_b$.
Noise	White and Gaussian with **two-sided** power spectral density $S_n(f) = N_0/2$. Projected onto an orthonormal basis it gives independent components, each $\mathcal{N}(0, N_0/2)$.
The Q function	$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt = \frac{1}{2}\operatorname{erfc}(x/\sqrt{2})$. It is the tail of a standard normal, so $Q(0) = \frac{1}{2}$ and $Q(-x) = 1 - Q(x)$.
sinc	Normalised: $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$, with zeros at every non-zero integer. Stated again wherever it is used.
Logarithms	$\log$ without a base means base two, and every entropy and every bit count is in bits. Decibels are $10\log_{10}$ of a power or energy ratio.
Signals	$s_i(t)$ is transmitted, $r(t)$ is received, $\{\psi_k(t)\}$ is the orthonormal basis, and boldface $\mathbf{s}_i$ is the point in signal space.
Priors	Symbols are equally likely unless a problem says otherwise, in which case the MAP form is used and the prior term written out.

THE TWO EXPRESSIONS THE WHOLE COURSE RESTS ON

$$P(\mathbf{s}_k \to \mathbf{s}_j) = Q\left(\sqrt{\frac{d_{kj}^2}{2N_0}}\right)$$

$$P_e \approx N_{\min}\, Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)$$

The first is exact for two signal points and is where every error probability in Chapters 2, 4 and 5 comes from. The second is its nearest-neighbour extension to M points, and the 2 in the denominator is the noise variance per dimension being $N_0/2$.

THE FOURIER TRANSFORM PAIR, AS CHAPTERS 1 AND 2 USE IT

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df$$

Written in hertz rather than in rad/s, so no factor of 2π appears in front of either integral. Sampling and the Nyquist criterion are both stated in this form.

PART

2

Summary of formulas

The results each chapter carries forward, in course order and without derivations.

A.1 The transition from analog to digital — Chapter 1

The results of Chapter 1, with $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$.

QUESTION	RESULT	SOURCE
What does sampling do to the spectrum?	$G_\delta(f) = f_s \sum_n G(f - nf_s)$: a copy at every multiple of f_s, scaled by f_s.	Ch. 1 · PS CH7.1.1
What is the lowest safe sampling rate?	The Nyquist rate $2W$. Below it the copies overlap and alias.	Ch. 1 · PS CH7.1.1
What filter turns the samples back into the signal?	An ideal lowpass filter with gain T_s. At $f_s = 2W$ its impulse response is $\mathrm{sinc}(2Wt)$.	Ch. 1 · PS CH7.1.1
How is $g(t)$ written in terms of its samples?	$g(t) = \sum_n g(nT_s)\,\mathrm{sinc}\big(2W(t - nT_s)\big)$, exact at every t.	Ch. 1 · PS CH7.1.1
Mid-rise or mid-tread: which has a level at zero?	Mid-tread. Mid-rise has a boundary at zero and outputs $\pm\Delta/2$ there.	Ch. 1 · PS CH7.2.1
What is the power of the quantization noise?	$E[Q^2] = \Delta^2/12$, with $\Delta = 2m_{\max}/L$, for a fine quantizer.	Ch. 1 · PS CH7.2.1
What does one more bit buy?	6.02 dB: $\mathrm{SQNR} = \alpha + 6.02R$, and $\alpha = 1.76$ dB for a full-scale sinusoid.	Ch. 1 · PS CH7.2.1
Why compand?	Small amplitudes are common. Finer steps near zero keep the SQNR steady as the level falls.	Ch. 1 · PS CH7.2.1
What bit rate does PCM need?	$R_b = R\,f_s$. Telephone speech: $8(8000) = 64$ kb/s.	Ch. 1 · PS CH7.3
Why use a Gray code?	Adjacent levels differ in one bit, so a small decision error costs one bit error.	Ch. 1 · PS CH7.3
What bandwidth does PCM need?	$B_T \geq R_b/2$, which is RW at $f_s = 2W$. Each extra bit adds 6.02 dB and W hertz.	Ch. 1 · PS CH7.4.1
What do DPCM and delta modulation send?	The difference from a prediction. Delta modulation sends one bit a sample and needs Δf_s above the largest slope.	Ch. 1 · PS CH7.4.2–7.4.3

A.2 Baseband transmission of digital signals — Chapter 2

The results of Chapter 2.

QUESTION	RESULT	SOURCE
Which filter gives the largest peak SNR?	The matched filter $h(t) = k\,g(T-t)$, or $H(f) = k\,G^*(f)\,e^{-j2\pi fT}$, sampled at $t = T$.	Ch. 2 · PS CH8.3.2
What is that largest peak SNR?	$\eta_{\max} = 2E/N_0$. It depends on the pulse energy E, not on its shape.	Ch. 2 · PS CH8.3.2
Where do the two symbols of polar signalling sit?	At $\pm\sqrt{E_b}$ on the axis of the unit-energy $\psi(t)$, a distance $2\sqrt{E_b}$ apart.	Ch. 2 · PS CH8.2.1
When do the correlator and the matched filter agree?	At $t = T_b$, where both give $\int_0^{T_b} x(t)\,\psi(t)\,dt$, for any shape of ψ.	Ch. 2 · PS CH8.3.1
What noise is left in the demodulator output?	$y = s_m + n$, with n Gaussian of mean 0 and variance $N_0/2$.	Ch. 2 · PS CH8.3.1
Where does the optimal threshold go?	$\lambda_{\text{opt}} = \frac{N_0}{4\sqrt{E_b}} \ln \frac{p_0}{p_1}$, at 0 for equal priors and toward the less likely symbol otherwise.	Ch. 2 · PS CH8.3.3
What is $Q(x)$?	The Gaussian tail $Q(x) = \frac{1}{2}\,\mathrm{erfc}(x/\sqrt{2})$, the chance that a unit Gaussian exceeds x.	Ch. 2 · PS CH8.3.3
What is P_b for polar signalling?	$P_b = Q\left(\sqrt{2E_b/N_0}\right)$ with equal priors and a matched filter.	Ch. 2 · PS CH8.3.3
How much interference does an RC channel add?	A bit m bits back adds $(1-q)q^m$, with $q = e^{-2\pi BT_b}$. The worst case totals q.	Ch. 2 · PS CH10.1.1
When does the eye close?	The opening is $2(1-2q)$. It closes at $BT_b = \ln 2/(2\pi) = 0.110$, and errors then occur without noise.	Ch. 2
What is Nyquist's criterion?	$p(kT_b) = \delta[k]$ exactly when $\sum_n P(f - nR_b) = T_b$. It needs $B_T \geq R_b/2$.	Ch. 2 · PS CH10.3.1
What bandwidth does a raised cosine need?	$B_T = \frac{R_b}{2}(1+\alpha)$. The tails fall as $1/\lvert t\rvert^3$ for $\alpha > 0$.	Ch. 2 · PS CH10.3.1

A.3 Geometric representation of signal waveforms — Chapter 3

The results of Chapter 3.

QUESTION	RESULT	SOURCE
What is an orthonormal set?	Functions with $\int \psi_j \psi_k \, dt = 1$ for $j = k$ and 0 for $j \neq k$: unit energy, and every pair orthogonal.	Ch. 3 · PS CH8.1
How is a coordinate computed?	$s_{ij} = \int_0^T s_i(t)\, \psi_j(t)\, dt$, one correlator for each basis function.	Ch. 3 · PS CH8.1
How is the waveform rebuilt?	$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \psi_j(t)$. The vector $\mathbf{s}_i$ holds the whole waveform.	Ch. 3 · PS CH8.1
What does the translation preserve?	Inner products: $\int x(t)\, y(t)\, dt = \sum_k x_k y_k$.	Ch. 3 · PS CH8.1
What is the energy of a signal?	$E_i = \|\mathbf{s}_i\|^2$, the squared distance from its point to the origin.	Ch. 3 · PS CH8.1
What is the distance between two signals?	$d_{ik} = \|\mathbf{s}_i - \mathbf{s}_k\|$, the root of $\int (s_i - s_k)^2 \, dt$.	Ch. 3 · PS CH8.1
Antipodal or orthogonal: which is further apart?	Antipodal, $d = 2\sqrt{E_b}$ against $\sqrt{2E_b}$. Orthogonal needs twice the energy, 3 dB, for the same distance.	Ch. 3 · PS CH8.2
When are a cosine and a sine orthogonal?	Over $[0, T]$ when $f_c T$ is a whole number. Scaled by $\sqrt{2/T}$ they are an orthonormal pair.	Ch. 3 · PS CH8.6.1
How far apart are the points of M-PSK?	$d_{\min} = 2\sqrt{E}\sin(\pi/M)$. All M points lie on the circle of radius $\sqrt{E}$.	Ch. 3 · PS CH8.6.1
What is one Gram–Schmidt step?	$g_k = s_k - \sum_{i<k} s_{ki}\psi_i$, then $\psi_k = g_k / \sqrt{E_{g_k}}$. A zero g_k adds no axis.	Ch. 3 · PS CH8.1
How many axes does a set of M signals need?	$N \leq M$, with $N = M$ only when no signal is a combination of the others.	Ch. 3 · PS CH8.1
What does another basis change?	Only the coordinates. N, the energies and the distances stay, and so do the receiver and its error probability.	Ch. 3 · PS CH8.1

A.4 The optimal receiver in additive white Gaussian noise — Chapter 4

The results of Chapter 4.

QUESTION	RESULT	SOURCE
How many bits does one of M waveforms carry?	$k = \log_2 M$, so $R_b = kR_s$ and $T_b = T/k$.	Ch. 4 · PS CH8.4
What does the correlator bank give?	$\mathbf{r} = \mathbf{s}_i + \mathbf{n}$, with $r_j = \int_0^T r(t)\,\psi_j(t)\,dt$.	Ch. 4 · PS CH8.4.1
When does a matched filter give the same number?	At the sampling time $t = T$. Its impulse response is $\psi_j(T-t)$.	Ch. 4 · PS CH8.4.1
Why can the noise outside the signal space be dropped?	It does not depend on the signal and adds the same $\|\|n'\|\|^2$ to every distance.	Ch. 4 · PS CH8.4.1
What are the noise components?	Independent Gaussians with mean 0 and variance $N_0/2$. The cloud is round.	Ch. 4 · PS CH8.4.1
What does MAP maximise?	$P(\mathbf{s}_i)\,f(\mathbf{r} \mid \mathbf{s}_i)$. With equal priors it is ML.	Ch. 4 · PS CH8.4.1
What is ML in Gaussian noise?	Minimum distance. MAP subtracts $N_0 \ln P(\mathbf{s}_i)$ from each squared distance.	Ch. 4 · PS CH8.4.1
What is the correlation metric?	$\mathbf{r} \cdot \mathbf{s}_i - E_i/2 + \frac{N_0}{2} \ln P(\mathbf{s}_i)$. Keep the largest.	Ch. 4 · PS CH8.4.1
What shape are the decision regions?	Convex polygons cut by bisectors. Unequal priors shift each line toward the less likely point.	Ch. 4 · PS CH8.4.1
What is the binary error probability?	$Q\left(\sqrt{d^2/2N_0}\right)$ for equal priors.	Ch. 4 · PS CH8.3.3, 8.4.2
What is the union bound?	$P_e \le \frac{1}{M} \sum_i \sum_{j \ne i} Q\left(\sqrt{d_{ij}^2/2N_0}\right)$. The intelligent bound keeps only faces.	Ch. 4 · PS CH8.4.2
What is the nearest-neighbour form?	$\bar{N}_{\min} Q\left(\sqrt{d_{\min}^2/2N_0}\right)$. Close at high SNR, but not a bound.	Ch. 4 · PS CH8.4.2

A.5 Digital modulation methods — Chapter 5

The results of Chapter 5.

QUESTION	RESULT	SOURCE
What does a carrier do to the spectrum?	$U(f) = \frac{1}{2}S(f - f_c) + \frac{1}{2}S(f + f_c)$. The band doubles to $2W$ and the energy halves.	Ch. 5 · PS CH8.5.1
What does the IQ modulator send?	$s(t) = I\cos(2\pi f_c t) - Q\sin(2\pi f_c t)$: one point of the plane a symbol.	Ch. 5 · PS CH8.6, 8.7
How do the binary schemes compare?	BPSK: $Q\left(\sqrt{2E_b/N_0}\right)$. BFSK and BASK: $Q\left(\sqrt{E_b/N_0}\right)$, 3 dB worse.	Ch. 5 · PS CH8.3.3, 8.6.1, 9.5
What is the M-PSK neighbour distance?	$d_{\min} = 2\sqrt{E_s}\sin(\pi/M)$, and $P_e \approx 2Q\left(\sqrt{2E_s/N_0}\sin(\pi/M)\right)$.	Ch. 5 · PS CH8.6.1, 8.6.3
What do Gray labels buy?	Neighbours differ in one bit, so $P_b \approx P_e/\log_2 M$.	Ch. 5 · PS CH8.6.1, 8.6.3
What does DPSK save, and what does it cost?	No carrier phase. $P_b = \frac{1}{2}e^{-E_b/N_0}$, under 1 dB behind BPSK.	Ch. 5 · PS CH8.6.4, 8.6.5
What is the M-ASK error?	$\frac{2(M-1)}{M}Q\left(\sqrt{6E_s/((M^2-1)N_0)}\right)$, about 6 dB a doubling.	Ch. 5 · PS CH8.5.3
What is the square-QAM error?	$4(1 - 1/\sqrt{M})\,Q\left(\sqrt{3E_s/((M-1)N_0)}\right)$, with $E_s = (M-1)d^2/6$.	Ch. 5 · PS CH8.7.3
How much does QAM save over PSK?	4.20 dB at $M = 16$ and 9.95 dB at $M = 64$.	Ch. 5 · PS CH8.7.3
What do M orthogonal signals give?	$M - 1$ neighbours at $\sqrt{2E_s}$. The E_b/N_0 needed falls as M grows, toward -1.6 dB.	Ch. 5 · PS CH9.1.1, 9.1.2
What does noncoherent BFSK need?	Tones $1/T$ apart and an envelope detector. $P_b = \frac{1}{2}e^{-E_b/2N_0}$.	Ch. 5 · PS CH9.5.2, 9.5.3
What is MSK?	BFSK at $\Delta f = 1/(2T_b)$ with a continuous phase. Its envelope is constant.	Ch. 5 · PS CH9.6.1, 9.6.2
How wide is each family?	PSK and QAM: $R_b/\log_2 M$. Orthogonal: $MR_b/(2\log_2 M)$.	Ch. 5 · PS CH10.2, 9.7
Where does each family sit on the plane?	PAM, PSK and QAM: bandwidth-limited, $r > 1$. Orthogonal: power-limited, $r < 1$.	Ch. 5 · PS CH9.7

A.6 An introduction to information theory — Chapter 6

The results of Chapter 6.

QUESTION	RESULT	SOURCE
How much information does a symbol carry?	$I(s_k) = -\log_2 p_k$ bits: rare symbols carry more.	Ch. 6 · PS CH12.1.1
What is entropy, and what bounds it?	$H(S) = -\sum_k p_k \log_2 p_k$, with $0 \le H \le \log_2 K$.	Ch. 6 · PS CH12.1.1
What does an extended source carry?	$H(S^n) = nH(S)$ for a memoryless source.	Ch. 6 · PS CH12.1.2
What are typical sequences?	About 2^{nH} sequences, each of probability near 2^{-nH}, that hold almost all the probability.	Ch. 6 · PS CH12.2
When do codeword lengths allow a prefix code?	Exactly when $\sum_k 2^{-l_k} \le 1$.	Ch. 6
How close can a code get to the entropy?	$H \le \bar{L} < H + 1$, and $H + 1/n$ for blocks of n.	Ch. 6 · PS CH12.3.1
How is a Huffman code built?	Merge the two least likely entries until one is left, then read each codeword back.	Ch. 6 · PS CH12.3.1
What does Lempel–Ziv need to know about the source?	Nothing. It parses the stream into new phrases, and its cost falls toward H as the stream grows.	Ch. 6 · PS CH12.3.2
What is a binary symmetric channel?	Each bit flips with probability p. Hard-decision BPSK gives $p = Q\left(\sqrt{2E_b/N_0}\right)$.	Ch. 6 · PS CH12.4
What is mutual information?	$I(X;Y) = H(X) - H(X \mid Y)$, symmetric and never negative.	Ch. 6 · PS CH12.1.3
What is the capacity of a channel?	$C = \max_{p(x)} I(X;Y)$: $1 - H_b(p)$ for the BSC, $1 - \epsilon$ for the erasure channel.	Ch. 6 · PS CH12.5
What does the channel coding theorem say?	Rates below C can be made reliable. Rates above it cannot.	Ch. 6 · PS CH12.5
What is the capacity of a bandlimited channel?	$C = W \log_2(1 + P/N_0 W)$ b/s, which levels off at $1.44\,P/N_0$ as W grows.	Ch. 6 · PS CH12.5.1
What is the Shannon limit?	$E_b/N_0 > \ln 2 = -1.59$ dB. Uncoded BPSK sits 11.2 dB above it.	Ch. 6 · PS CH12.6

PART

3

Notation

Every symbol the course defines, with the chapter that defines it. A symbol is never reused for a second meaning.

$f_s,\ T_s$
: Sampling rate in hertz and the interval between samples, $T_s = 1/f_s$.

W
: The highest frequency present in a message. The sampling theorem asks for $f_s > 2W$.

$\mathrm{sinc}(x)$
: Normalised: $\sin(\pi x)/(\pi x)$, with zeros at every non-zero integer.

$\Delta,\ L,\ n$
: Quantizer step, number of levels and bits a sample, with $L = 2^n$ and $\Delta = 2V/L$ over a range $\pm V$.

σ_q^2
: Quantization noise power, $\Delta^2/12$ for a uniform quantizer.

$T,\ R$
: Symbol interval and the rate that follows from it. A bit interval is written T_b and a bit rate R_b.

$p(t),\ h(t)$
: The transmitted pulse and the impulse response of the receive filter. The matched filter is $h(t) = p(T - t)$.

α
: Roll-off of a raised-cosine pulse, from 0 to 1. Bandwidth is $(1 + \alpha)/2T$.

$\psi_k(t)$
: The k-th orthonormal basis function, from the Gram–Schmidt procedure applied to the signal set.

$s_i(t),\ \mathbf{s}_i$
: The i-th transmitted waveform, and the point in signal space whose coordinates are its projections onto the basis.

$E,\ E_i$
: Signal energy, normalised so that no resistance appears: $E_i = \|\mathbf{s}_i\|^2$.

$d_{ij},\ d_{\min}$
: Distance between two signal points, and the smallest such distance in a constellation.

$r(t),\ \mathbf{r}$
: The received waveform and the observation vector it produces, $\mathbf{r} = \mathbf{s}_i + \mathbf{n}$.

N_0
: Noise power spectral density parameter. The two-sided density is $N_0/2$ watts per hertz, and each noise component is $\mathcal{N}(0, N_0/2)$.

$Q(x)$
: The Gaussian tail: the probability that a standard normal variable exceeds x. Equal to $\frac{1}{2}\,\mathrm{erfc}(x/\sqrt{2})$.

$M,\ N$
: The number of signals in the set, and the number of basis functions they need. M signals carry $\log_2 M$ bits a symbol.

$N_{\min}$
: The average number of signal points at the minimum distance. An average, so not usually a whole number.

$P_b,\ P_e,\ P_s$
: Bit error probability, error probability in general, and symbol error probability.

$E_b,\ E_s$
: Energy per information bit and energy per transmitted symbol, with $E_s = (\log_2 M)E_b$.

f_c
: Carrier frequency. Every scheme in Chapter 5 changes the amplitude, the phase or the frequency of $\cos(2\pi f_c t)$.

$S,\ K,\ p_k$
: The source alphabet, how many symbols it has, and the probability of the k-th.

$I(s_k)$
: Self-information of one symbol, $-\log_2 p_k$ bits.

$H(S)$
: Entropy: the average information a symbol carries, in bits a symbol. Bounded by 0 and $\log_2 K$.

$\bar{L},\ l_k$
: Average codeword length and the length of the codeword for symbol k.

η
: Coding efficiency, $H(S)/\bar{L}$. Never above one.

σ^2
: Variance of the codeword length about $\bar{L}$. Separates two Huffman codes that share an average.

$p(y_k \mid x_j)$
: Transition probability of a discrete memoryless channel. Collected into the channel matrix, whose every row sums to one.

$H(X \mid Y)$
: Conditional entropy: the uncertainty still remaining about the input after the output has been seen.

$I(X;Y)$
: Mutual information, $H(X) - H(X \mid Y)$: what the channel actually delivered, in bits per use. Symmetric and never negative.

C
: Channel capacity, the largest $I(X;Y)$ over all input distributions. For the binary symmetric channel, $1 - H(p)$.

$H(p)$
: The binary entropy function, $-p\log_2 p - (1-p)\log_2(1-p)$. It measures a two-symbol source and also the noise a binary symmetric channel adds.

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VERSION	DATE	CHANGES
v0	24 September 2026	First edition.