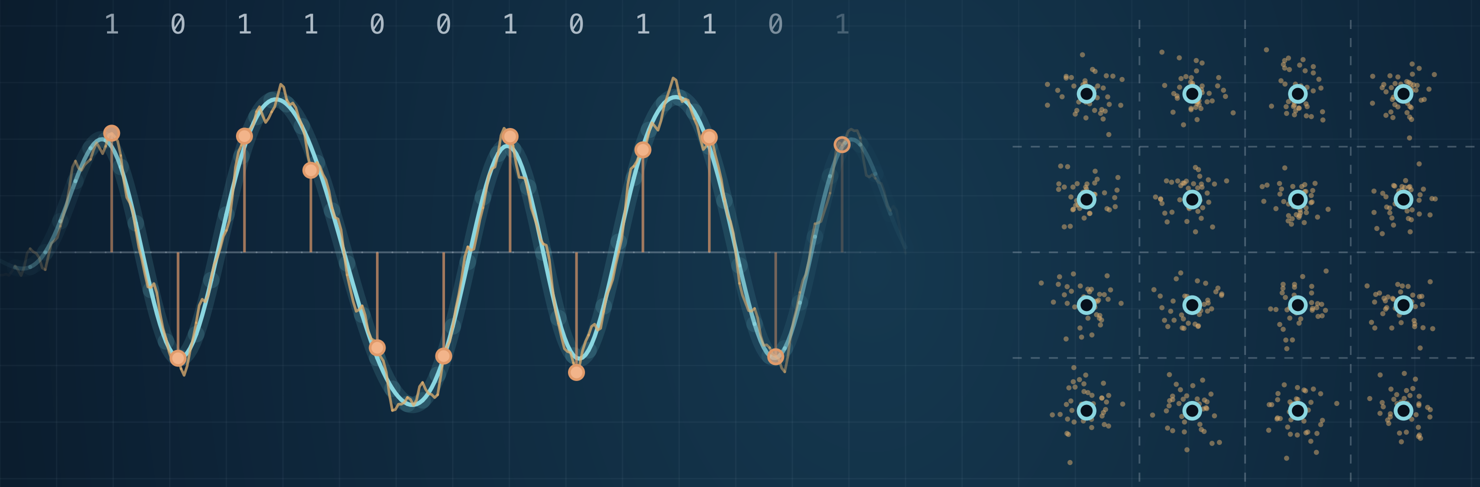
SAMPLING · DETECTION · MODULATION · CODING

Digital Communications

Student Workbook



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180 QUESTIONS · MODULES 1–6

Contents

Every question in the course, with no answer and no solution. Work each one on the page, then check it against the artifact or against the instructor edition.

How to use it

The questions are in the order the course meets them, and each is labelled with what it asks for. Only the statement and its lettered parts are printed; the reasoning stays for you to supply. The question numbers are shared with every other edition, so D5-04 is the same question in the artifact, in this workbook and in the instructor solutions.

MODULE

1

Sampling, Quantization and PCM

30 questions on sampling, quantization and pcm. Write your reasoning in the space under each one.

D1-01 · PCM DESIGN FROM AN ACCURACY REQUIREMENT

Assume that a sinusoidal message signal is defined as $x(t) = V_{\max}\cos(12000\pi t)$, where $V_{\max}$ is the maximum amplitude of the message signal. This analog message signal is sampled at the Nyquist rate and quantized by using a uniform quantizer. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$, where Δ is the step size. The quantization noise is required not to exceed $\pm 0.25\%$ of the peak-to-peak message signal. Quantized data are encoded by using a 4-level PAM system.

a) What is the minimum number of bits per sample for this PAM system?

b) Calculate the bit rate of this system.

c) What is the symbol rate of this system?

D1-02 · PCM DESIGN FROM AN ACCURACY REQUIREMENT

Assume that a sinusoidal message signal is defined as $x(t) = V_{\max}\sin(7000\pi t)$, where $V_{\max}$ is the maximum amplitude of the message signal. It is sampled at a rate 25% greater than the Nyquist rate and quantized by using a uniform quantizer. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$. It is required not to exceed $\pm 0.8\%$ of the peak-to-peak message signal. Quantized data are encoded by using an 8-level PAM system.

a) What is the minimum number of bits per sample?

b) Determine the sampling rate.

c) Calculate the bit rate of this system.

d) What is the symbol rate of this system?

D1-03 · PCM DESIGN FROM AN ACCURACY REQUIREMENT

Let $X(t)$ have a bandwidth of 3.2 MHz. This signal is sampled, quantized and binary encoded to obtain a PCM signal.

a) Determine the sampling rate if $X(t)$ is sampled at a rate 25% greater than the Nyquist rate.

b) If the samples of $X(t)$ are quantized by using a uniform quantizer with 2048 levels, determine the number of bits required per sample.

c) Calculate the bit rate of this system in bits per second.

d) Find the least channel bandwidth that can carry this PCM signal.

D1-04 · PCM DESIGN FROM AN ACCURACY REQUIREMENT

A PCM system carries the sinusoid $x(t) = V_{\max}\cos(10000\pi t)$. It samples at the Nyquist rate and quantizes uniformly over $[-V_{\max}, V_{\max}]$. The quantization noise is uniform between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$. The bits are sent by a 16-level PAM system, and the measured symbol rate is $17\,500$ symbols per second.

a) Find the bit rate and the number of bits per sample.

b) Find the number of quantization levels and the step size in terms of $V_{\max}$.

c) Find the largest quantization error as a percentage of the peak-to-peak message signal.

d) The design came from a requirement of $\pm p\%$ of the peak-to-peak signal. Find the range of p for which this number of bits is the minimum.

D1-05 · PCM DESIGN FROM AN ACCURACY REQUIREMENT

Assume that a sinusoidal message signal is defined as $x(t) = V_{\max}\cos(18000\pi t)$. It is sampled at the Nyquist rate and quantized by using a uniform quantizer. The quantization noise is uniform between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$. It is required not to exceed $\pm 0.1\%$ of the peak-to-peak message signal. The bits are sent by an M-level PAM system over a channel that accepts at most $60\,000$ symbols per second.

a) What is the minimum number of bits per sample?

b) Calculate the bit rate of this system.

c) Find the smallest PAM order M, a power of two, that the channel accepts.

d) Calculate the symbol rate with that M.

D1-06 · PCM DESIGN FROM AN ACCURACY REQUIREMENT

Let $X(t)$ have a bandwidth of 2.5 MHz. It is sampled at a rate 30% greater than the Nyquist rate, quantized by a uniform quantizer and binary encoded. The PCM signal must fit a link that carries 64 Mbit/s. Assume that 1 Mbit/s $= 10^6$ bit/s.

a) Determine the sampling rate.

b) Find the largest number of bits per sample, and the number of levels, that the link allows.

c) Calculate the bit rate of the resulting system.

d) For a full-scale sinusoidal test signal, calculate the SQNR in dB.

D1-07 · THE SPECTRUM OF A PRODUCT, A SQUARE OR A SINC

Let $x(t) = 10\cos(3000\pi t)\cos(6000\pi t)$ be sampled and quantized by using a 512-level uniform quantizer. Assume that 1 kbit/s $= 1000$ bit/s.

a) Determine the minimum sampling rate if a guard band of 1 kHz is required.

b) Calculate the bit rate of this system.

c) If the data rate of this system is required as 108 kbit/s, what should the guard band be?

d) Calculate the step size of the uniform quantizer.

D1-08 · THE SPECTRUM OF A PRODUCT, A SQUARE OR A SINC

Let $x(t) = 5\sin(2000\pi t)\cos(8000\pi t)$ be sampled and quantized by using a 64-level uniform quantizer. Assume that 1 kbit/s $= 1000$ bit/s.

a) Determine the minimum sampling rate if a guard band of 1.5 kHz is required.

b) Calculate the bit rate of this system.

c) If the data rate of this system is required as 78 kbit/s, what should the guard band be?

d) Calculate the step size of the uniform quantizer.

D1-09 · THE SPECTRUM OF A PRODUCT, A SQUARE OR A SINC

Let $x(t) = 8\cos^2(3000\pi t)$ be sampled and quantized by using a 256-level uniform quantizer that spans the range of $x(t)$. Assume that 1 kbit/s $= 1000$ bit/s.

a) Determine the minimum sampling rate if a guard band of 2 kHz is required.

b) Calculate the bit rate of this system.

c) If the data rate of this system is required as 72 kbit/s, what should the guard band be?

d) Calculate the step size of the uniform quantizer.

D1-10 · THE SPECTRUM OF A PRODUCT, A SQUARE OR A SINC

The signal

$$x(t) = \left(\frac{\sin(200\pi t)}{20\pi t}\right)^2$$

is sampled at the Nyquist rate. The samples are uniformly quantized with 256 levels over the range of $x(t)$. Use $\mathrm{sinc}(u) = \sin(\pi u)/(\pi u)$. According to the information given above,

a) Calculate the bit rate of this system.

b) Calculate the step size of the uniform quantizer.

c) Find the quantized value and the natural binary code word of the samples at $t = 0$, $t = 2.5$ ms and $t = 7.5$ ms.

D1-11 · THE SPECTRUM OF A PRODUCT, A SQUARE OR A SINC

The signal

$$x(t) = \frac{\sin(200\pi t)}{10\pi t} \cdot \frac{\sin(600\pi t)}{10\pi t}$$

is sampled at the Nyquist rate. The samples are uniformly quantized with 128 levels over the range of $x(t)$. Use $\mathrm{sinc}(u) = \sin(\pi u)/(\pi u)$. According to the information given above,

a) Calculate the bit rate of this system.

b) Calculate the step size of the uniform quantizer. (Hint: $\min x(t) \cong -177$.)

c) Find the smallest number of levels, a power of two, that makes the step size smaller than 1. Calculate the bit rate it needs.

D1-12 · THE SPECTRUM OF A PRODUCT, A SQUARE OR A SINC

The signal

$$x(t) = \frac{\sin(1000\pi t)}{\pi t} \cos(4000\pi t)$$

is sampled at the Nyquist rate. The samples are uniformly quantized with 1024 levels over the range of $x(t)$. According to the information given above,

a) Find the Fourier transform $X(f)$ and the highest frequency in $x(t)$.

b) Calculate the bit rate of this system.

c) Calculate the step size of the uniform quantizer. (Hint: $\min x(t) \cong -902$.)

d) Calculate the energy of $x(t)$ by Parseval's theorem.

D1-13 · A SUM OF SINUSOIDS THROUGH A UNIFORM QUANTIZER

The signal $x(t) = 2\cos(3000\pi t) + 3\cos(9000\pi t)$ is sampled at the Nyquist rate and samples are uniformly quantized with 256 levels. According to the information given above,

a) Calculate the bit rate of this system.

b) Calculate the step size of the uniform quantizer.

c) Calculate the SQNR of the quantization scheme (in dB).

D1-14 · A SUM OF SINUSOIDS THROUGH A UNIFORM QUANTIZER

The signal $x(t) = 2\cos(2000\pi t) + \cos(4000\pi t)$ is sampled at the Nyquist rate and samples are uniformly quantized with 128 levels. The quantizer spans the range of $x(t)$, from its minimum to its maximum. According to the information given above,

a) Calculate the bit rate of this system.

b) Find the maximum and the minimum of $x(t)$.

c) Calculate the step size of the uniform quantizer.

d) Calculate the SQNR of the quantization scheme (in dB).

D1-15 · A SUM OF SINUSOIDS THROUGH A UNIFORM QUANTIZER

The signal $x(t) = \cos(1000\pi t) + \cos(3000\pi t) + \cos(5000\pi t)$ is sampled at the Nyquist rate and samples are uniformly quantized with 512 levels. According to the information given above,

a) Calculate the bit rate of this system.

b) Calculate the step size of the uniform quantizer.

c) Calculate the SQNR of the quantization scheme (in dB).

D1-16 · A SUM OF SINUSOIDS THROUGH A UNIFORM QUANTIZER

The signal $x(t) = 3\sin(5000\pi t) + 4\cos(5000\pi t)$ is sampled at the Nyquist rate and samples are uniformly quantized with 128 levels over the range of $x(t)$. According to the information given above,

a) Calculate the bit rate of this system.

b) Find the peak value of $x(t)$.

c) Calculate the step size of the uniform quantizer.

d) Calculate the SQNR of the quantization scheme (in dB).

D1-17 · A SUM OF SINUSOIDS THROUGH A UNIFORM QUANTIZER

The signal $x(t) = 4\cos(2000\pi t) + 2\cos(6000\pi t)$ is sampled with a guard band of at least 1 kHz and uniformly quantized over the range of $x(t)$. The bits must fit a link of 80 kbit/s. Assume that 1 kbit/s $=$ 1000 bit/s.

a) Determine the minimum sampling rate.

b) Find the largest number of bits per sample that the link allows at that rate.

c) Calculate the step size of the uniform quantizer.

d) Calculate the SQNR of the quantization scheme (in dB).

D1-18 · A DENSITY WITH A CONSTANT AND A FINE UNIFORM QUANTIZER

A strict-sense stationary random process $X(t)$ is sampled. The sampled values X have the following PDF: $f_X(x) = k\left[1 + x^2\right]$ for $x \in [-1, 1]$. Sampled values are quantized by using a uniform quantizer with 128 levels. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$, where Δ is the step size.

a) Determine the value of k.

b) Obtain the SQNR in dB.

c) If the bandwidth of the signal is 4 kHz, what is the bit rate of the corresponding PCM system?

D1-19 · A DENSITY WITH A CONSTANT AND A FINE UNIFORM QUANTIZER

A strict-sense stationary random process $X(t)$ is sampled. The sampled values X have the following PDF: $f_X(x) = k\left[1 - |x|^{1/2}\right]$ for $x \in [-1, 1]$. Sampled values are quantized by using a uniform quantizer with 512 levels. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$, where Δ is the step size.

a) Determine the value of k.

b) Obtain the SQNR in dB.

c) If the bandwidth of the signal is 5 kHz, what is the bit rate of the corresponding PCM system?

D1-20 · A DENSITY WITH A CONSTANT AND A FINE UNIFORM QUANTIZER

The samples of a stationary source, $X(t)$, are distributed according to the PDF $f_X(x) = k\left(4 - x^2\right)$ for $|x| \leq 2$, and zero elsewhere. The samples are quantized by using a uniform quantizer with 64 levels over $[-2, 2]$. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$.

a) Determine the value of k.

b) Obtain the SQNR in dB.

c) If the bandwidth of the signal is 6 kHz, what is the bit rate of the corresponding PCM system?

D1-21 · A DENSITY WITH A CONSTANT AND A FINE UNIFORM QUANTIZER

The samples of a stationary source $X(t)$ have the PDF $f_X(x) = k\,e^{-|x|}$ for $|x| \leq 2$, and zero elsewhere. They are quantized by a uniform quantizer with 256 levels over $[-2, 2]$. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$.

a) Determine the value of k.

b) Calculate the power of the samples.

c) Obtain the SQNR in dB.

d) Find the smallest number of levels, a power of two, that gives an SQNR of at least 50 dB.

D1-22 · A DENSITY WITH A CONSTANT AND A FINE UNIFORM QUANTIZER

The samples of a stationary source $X(t)$ have the PDF $f_X(x) = k(1+x)$ for $-1 \leq x \leq 1$, and zero elsewhere. The samples are quantized by a uniform quantizer with 32 levels over $[-1, 1]$. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$.

a) Determine the value of k.

b) Calculate the mean and the power of the samples.

c) Obtain the SQNR in dB.

d) If the bandwidth of the signal is 3.5 kHz, what is the bit rate of the corresponding PCM system?

D1-23 · A DENSITY WITH A CONSTANT AND A FINE UNIFORM QUANTIZER

The samples of a stationary source $X(t)$ have the PDF $f_X(x) = k|x|$ for $|x| \leq a$, and zero elsewhere. The power of the samples is $E[X^2] = 2$. A uniform quantizer with L levels covers $[-a, a]$. The quantization noise is a uniform random variable between $-\frac{\Delta}{2}$ and $\frac{\Delta}{2}$.

a) Determine a and k.

b) Find the smallest L, a power of two, that gives an SQNR of at least 40 dB.

c) Calculate the step size and the SQNR obtained with that L.

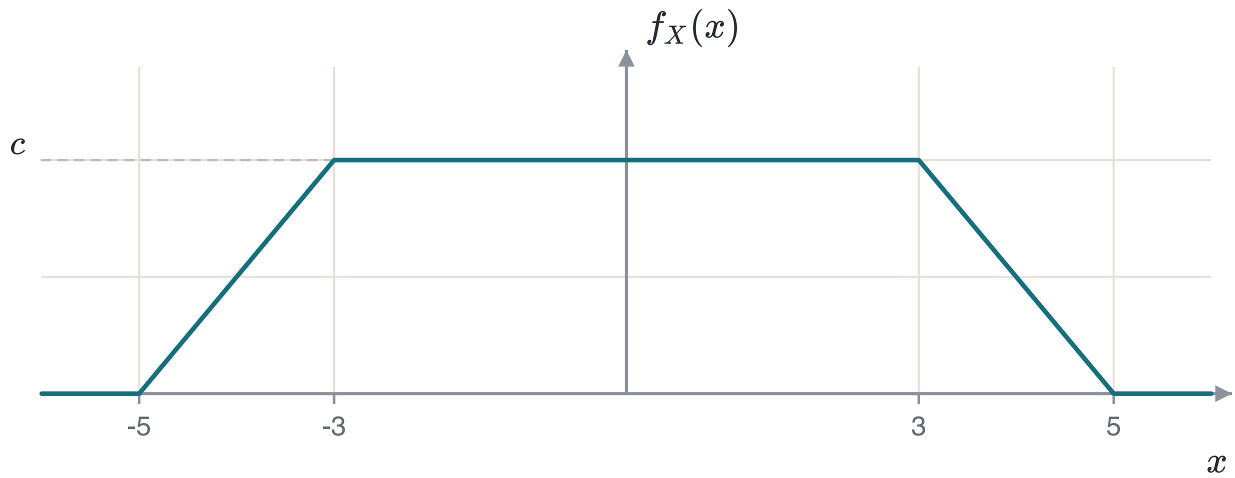
d) If the bandwidth of the signal is 6.5 kHz, what is the bit rate of the corresponding PCM system?

D1–24 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a stationary source, $X(t)$, are distributed according to the probability density function (PDF) drawn below. These samples are quantized using the following quantizer:

$$\hat{X} = \mathbb{Q}(X) = \begin{cases} -2, & -3 < X < 0 \\ 2, & 0 < X < 3 \\ 0, & \text{otherwise} \end{cases}$$

According to the information given above,



a) Determine the value of c.

b) Calculate the power of the samples of the stationary source.

c) Calculate the power of the quantization noise.

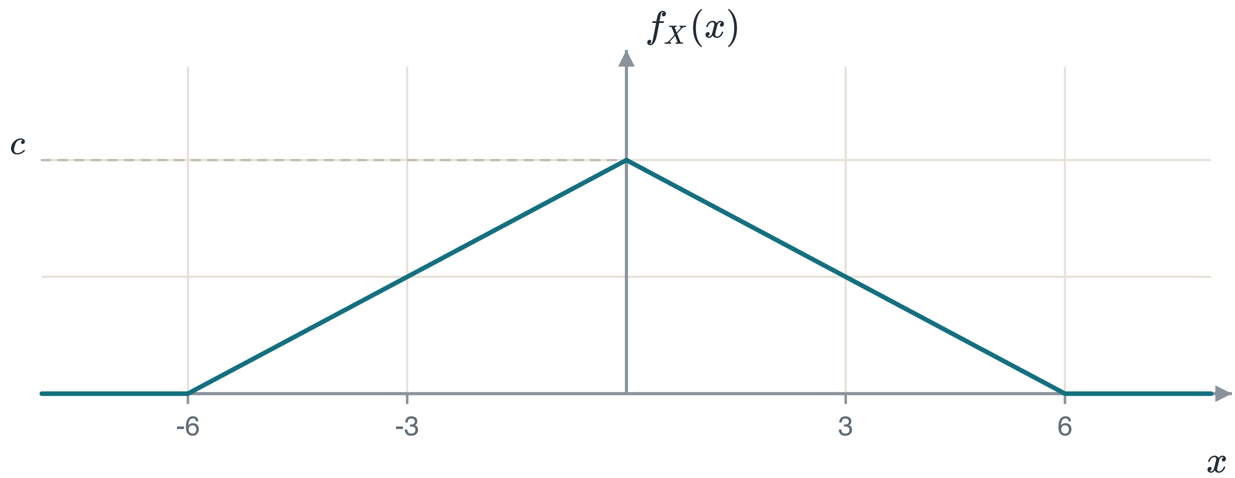
d) Obtain the SQNR in dB.

D1-25 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a stationary source, $X(t)$, are distributed according to the probability density function (PDF) drawn below. These samples are quantized using the following quantizer:

$$\hat{X} = \mathbb{Q}(X) = \begin{cases} -3, & -6 < X < 0 \\ 3, & 0 < X < 6 \\ 0, & \text{otherwise} \end{cases}$$

According to the information given above,



a) Determine the value of c.

b) Calculate the power of the samples of the stationary source.

c) Calculate the power of the quantization noise.

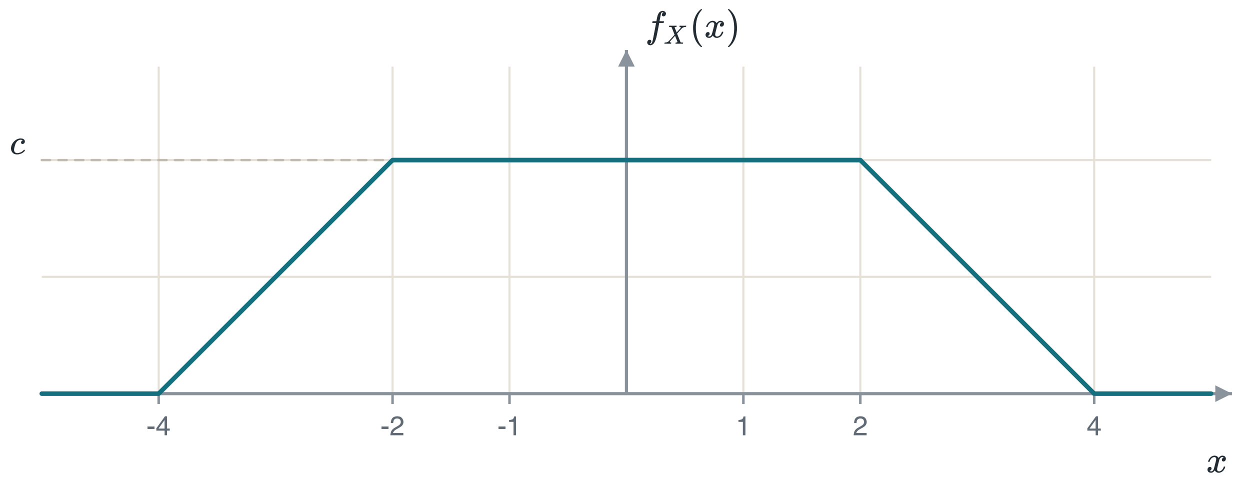
d) Obtain the SQNR in dB.

D1–26 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a stationary source, $X(t)$, are distributed according to the probability density function (PDF) drawn below. These samples are quantized using the following quantizer:

$$\hat{X} = \mathbb{Q}(X) = \begin{cases} -2, & -4 < X < -1 \\ 0, & -1 \leq X \leq 1 \\ 2, & 1 < X < 4 \end{cases}$$

According to the information given above,



a) Determine the value of c.

b) Calculate the power of the samples of the stationary source.

c) Calculate the power of the quantization noise.

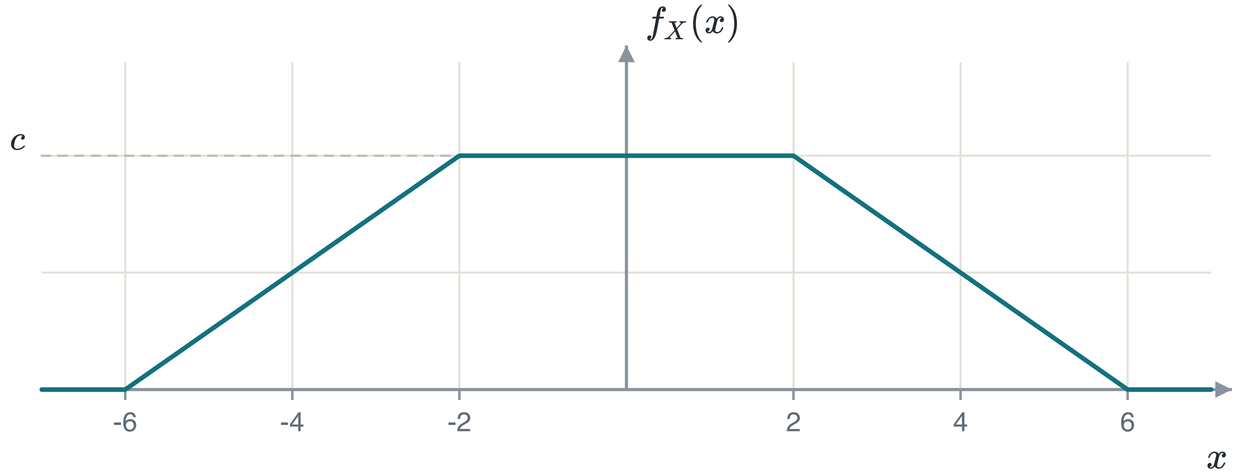
d) Obtain the SQNR in dB.

D1-27 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a stationary source, $X(t)$, are distributed according to the probability density function (PDF) drawn below. These samples are quantized using the following quantizer:

$$\hat{X} = \mathbb{Q}(X) = \begin{cases} -2, & -4 < X < 0 \\ 2, & 0 < X < 4 \\ 0, & \text{otherwise} \end{cases}$$

According to the information given above,



a) Determine the value of c.

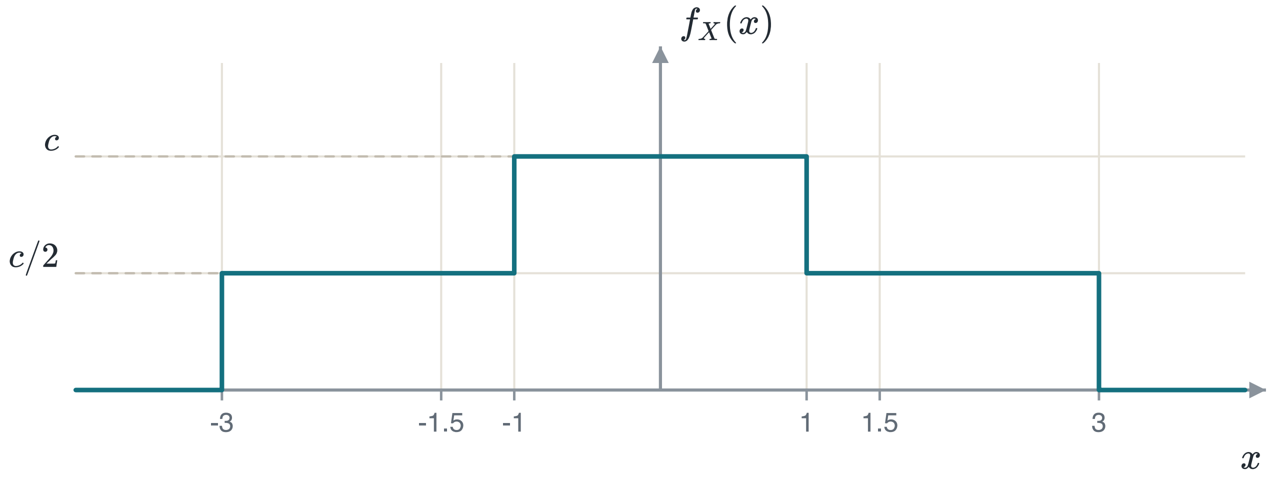
b) Calculate the power of the samples of the stationary source.

c) Calculate the power of the quantization noise.

d) Obtain the SQNR in dB.

D1-28 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a stationary source $X(t)$ have the piecewise-constant PDF drawn below. They are quantized by a 4-level uniform quantizer over $[-3,3]$, with outputs ± 0.75 and ± 2.25 and boundaries 0 and ± 1.5.



a) Determine the value of c.

b) Calculate the power of the samples.

c) Calculate the power of the quantization noise, region by region.

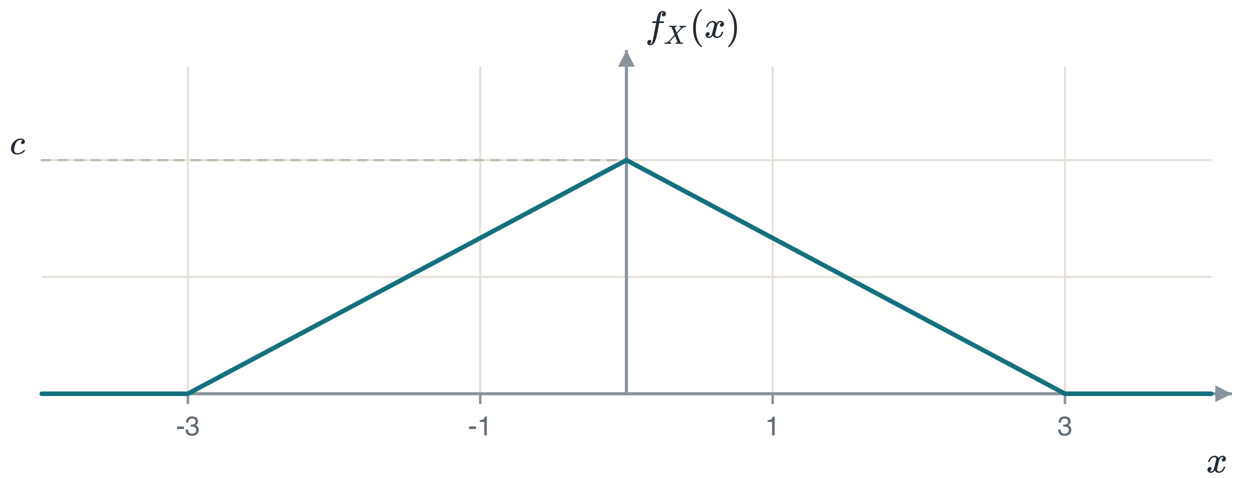
d) Obtain the SQNR in dB, and compare the noise power with $\Delta^2/12$.

D1–29 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a stationary source $X(t)$ have the triangular PDF drawn below. They are quantized by

$$\hat{X} = \mathbb{Q}(X) = \begin{cases} -b, & -3 < X < 0 \\ b, & 0 < X < 3 \\ 0, & \text{otherwise} \end{cases}$$

where $b > 0$ is a design constant.



a) Determine the value of c.

b) Calculate the power of the samples.

c) For $b = 2$, calculate the power of the quantization noise and the SQNR in dB.

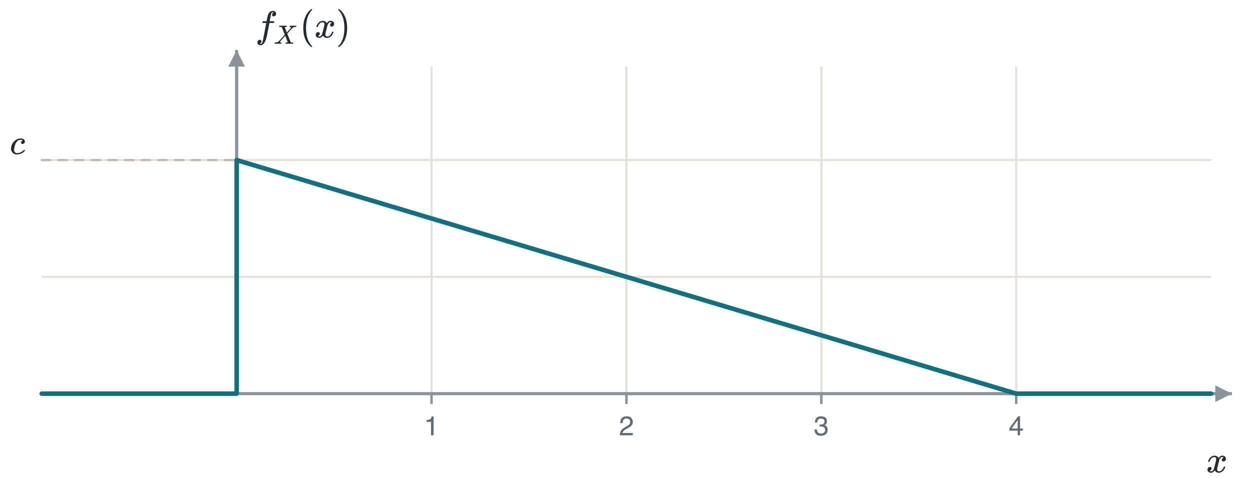
d) Find the value of b that makes the noise power smallest, and the SQNR it gives.

D1-30 · A DRAWN DENSITY AND A COARSE QUANTIZER

The samples of a non-negative stationary source $X(t)$ have the PDF drawn below. It falls linearly from c at $x = 0$ to zero at $x = 4$. They are quantized by

$$\hat{X} = \mathbb{Q}(X) = \begin{cases} 1, & 0 < X < 2 \\ 3, & 2 < X < 4 \\ 0, & \text{otherwise} \end{cases}$$

According to the information given above,



a) Determine the value of c.

b) Calculate the mean and the power of the samples.

c) Calculate the power of the quantization noise.

d) Obtain the SQNR in dB, and compare the noise power with $\Delta^2/12$.

MODULE

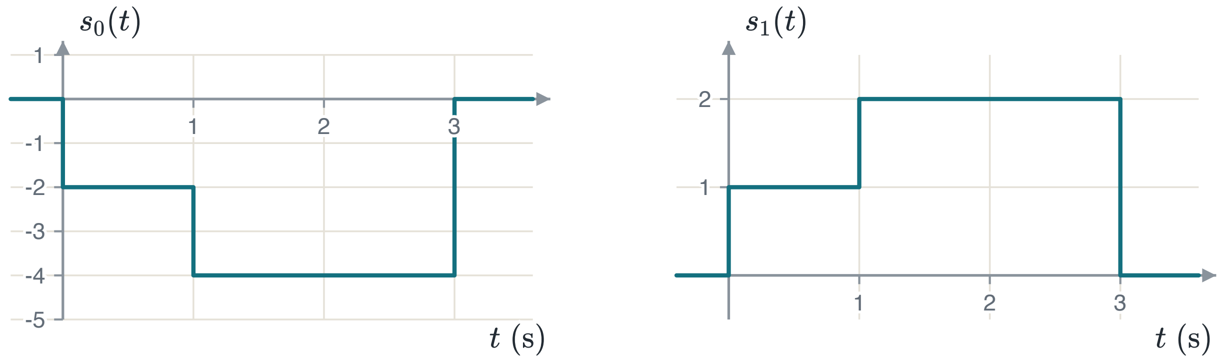
2

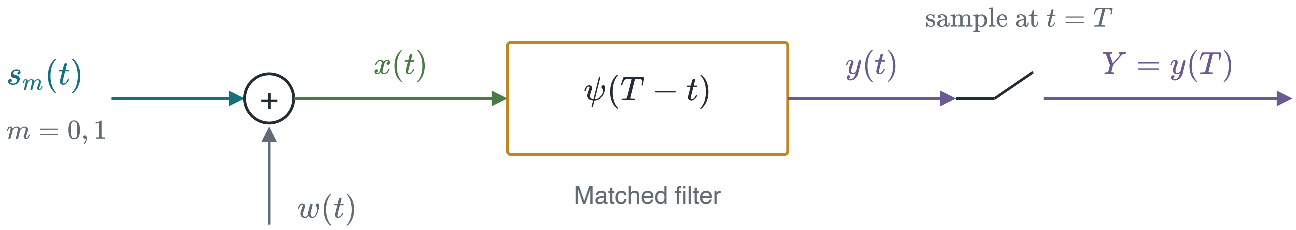
Baseband Transmission

30 questions on baseband transmission. Write your reasoning in the space under each one.

D2-01 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 2.25$ W/Hz. Two equiprobable messages, $s_0(t)$ for "0" and $s_1(t)$ for "1", are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 3$ s. According to the information given above,





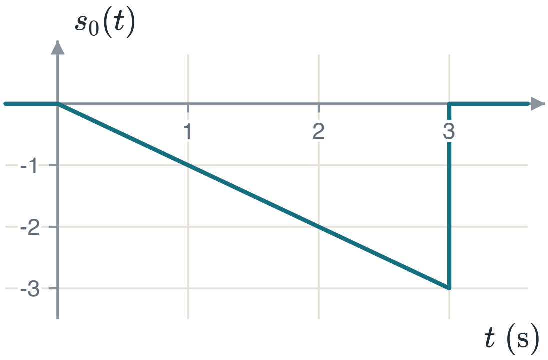
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

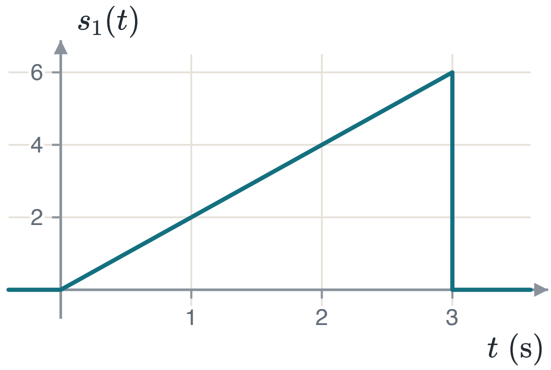
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

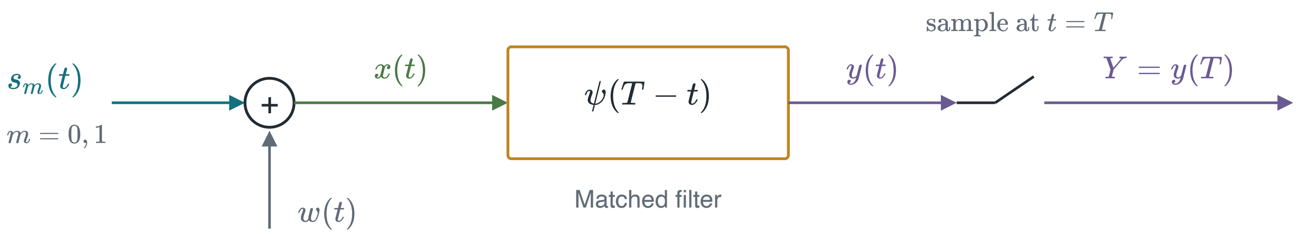
c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-02 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 4$ W/Hz. Two equiprobable messages, $s_0(t)$ for "0" and $s_1(t)$ for "1", are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 3$ s. According to the information given above,







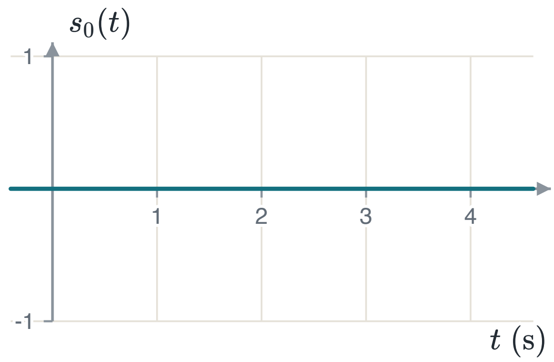
a) Plot the impulse response of the matched filter, $h(t) = \psi(T-t)$.

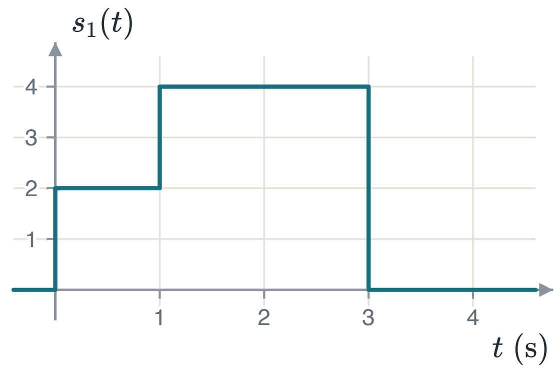
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

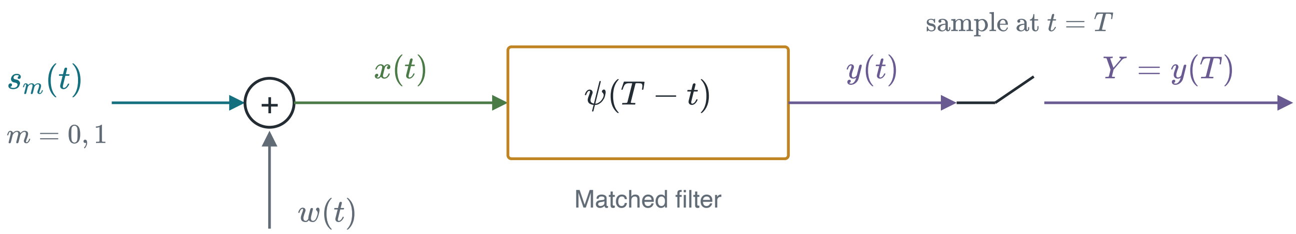
c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-03 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 4$ W/Hz. Two equiprobable messages, $s_0(t)$ for “0” and $s_1(t)$ for “1”, are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 4$ s. According to the information given above,







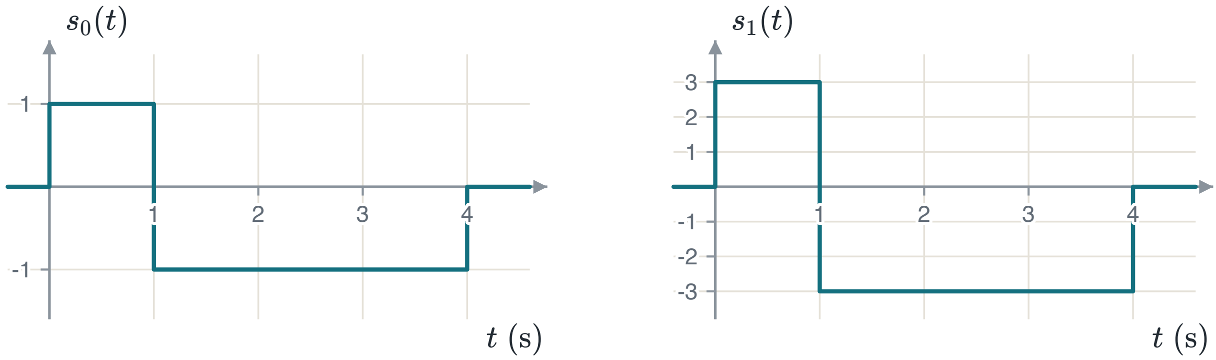
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-04 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 1$ W/Hz. Two equiprobable messages, $s_0(t)$ for "0" and $s_1(t)$ for "1", are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 4$ s. According to the information given above,



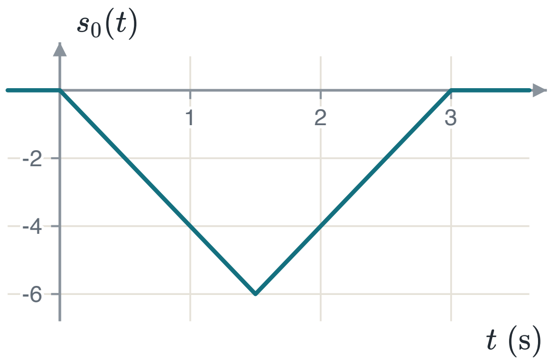
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

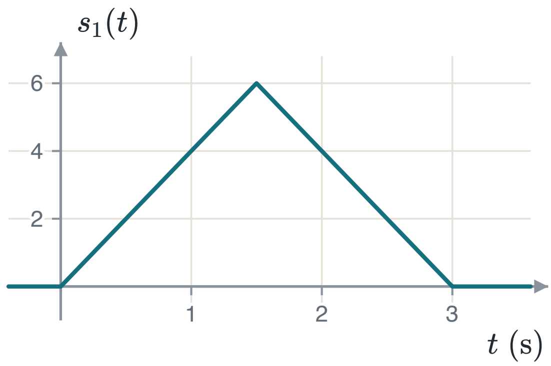
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

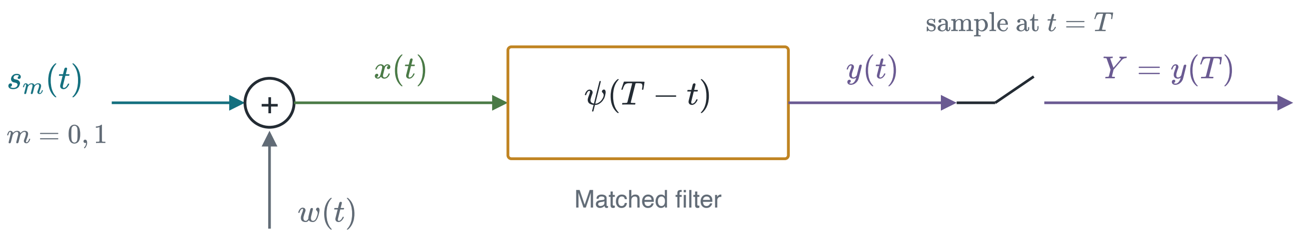
c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-05 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 6.25$ W/Hz. Two equiprobable messages, $s_0(t)$ for “0” and $s_1(t)$ for “1”, are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 3$ s. According to the information given above,







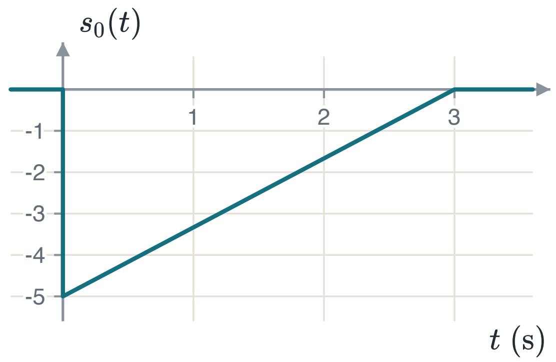
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-06 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 4$ W/Hz. Two equiprobable messages, $s_0(t)$ for “0” and $s_1(t)$ for “1”, are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 3$ s. According to the information given above,



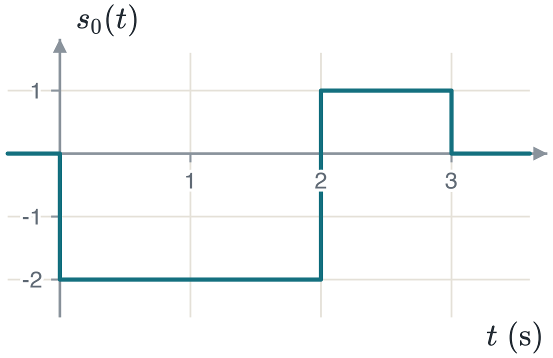
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

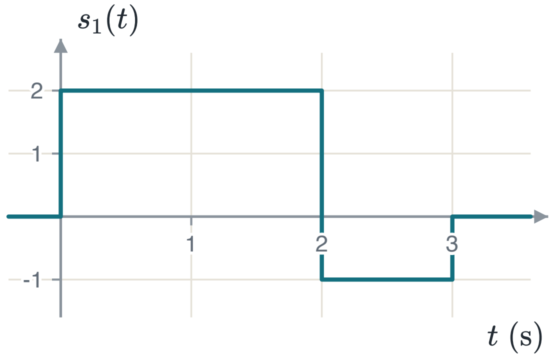
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

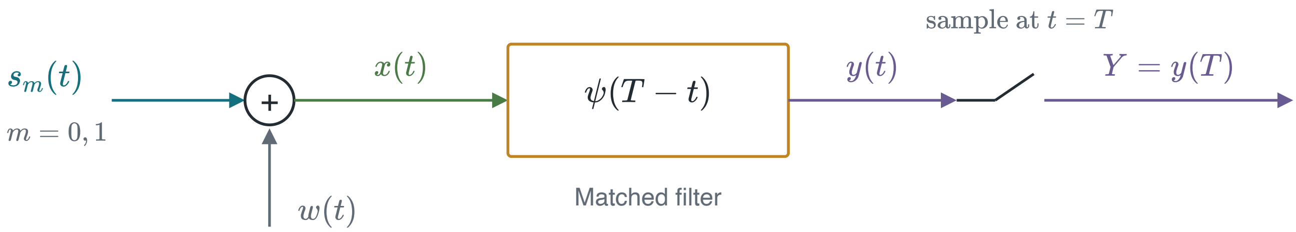
c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-07 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 2.25$ W/Hz. Two messages are transmitted by the waveforms below: $s_0(t)$ for "0" and $s_1(t)$ for "1". The message "0" is sent with probability 0.8 and "1" with probability 0.2. The signals pass through the matched-filter type demodulator shown, with the unit-energy basis signal $\psi(t)$ and $T = 3$ s. According to the information given above,







a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

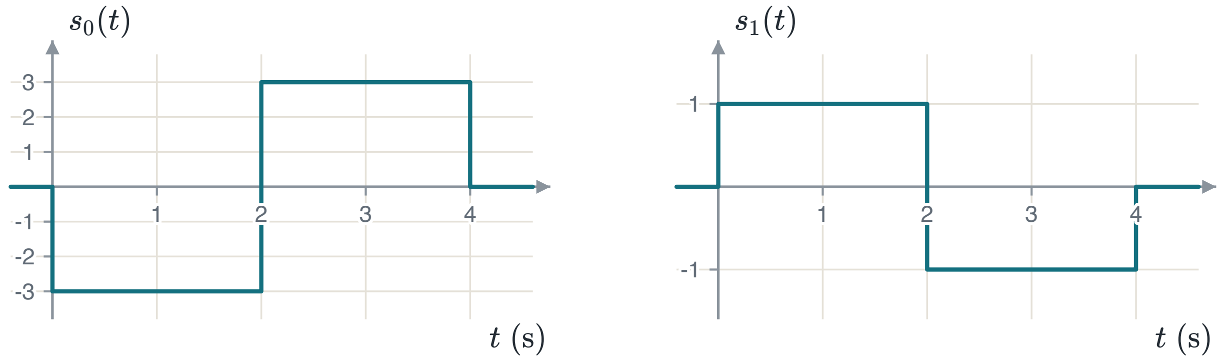
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold (λ).

d) Calculate P_b at this threshold. Compare it with P_b for the threshold $\lambda = 0$.

D2-08 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

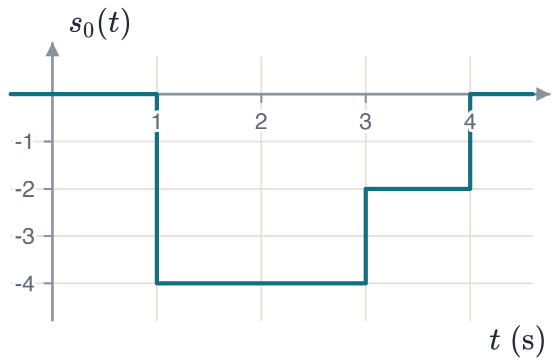
Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 2.56$ W/Hz. Two equiprobable messages, $s_0(t)$ for "0" and $s_1(t)$ for "1", are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 4$ s. According to the information given above,

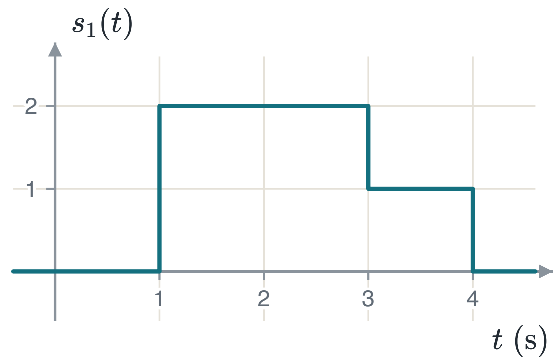


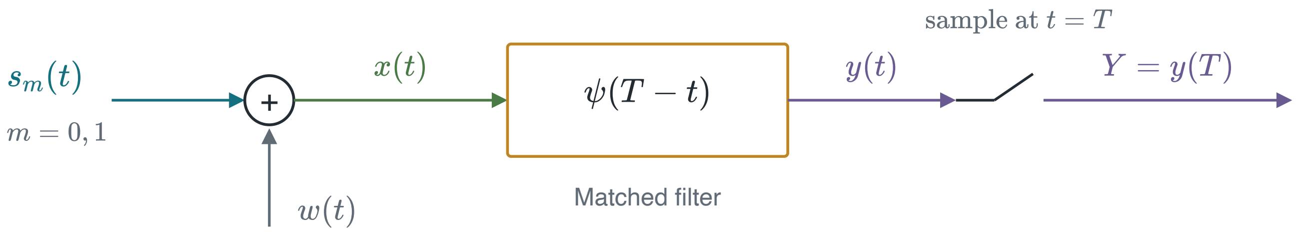
- **a)** Show that one basis signal represents both waveforms. Give $\psi(t)$ and the coordinates of s_0 and s_1.
- **b)** Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.
- **c)** Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.
- **d)** Find the optimal decision threshold (λ) and calculate P_b with a table of the Q function.

D2-09 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 6.25$ W/Hz. Two equiprobable messages, $s_0(t)$ for “0” and $s_1(t)$ for “1”, are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 4$ s. According to the information given above,







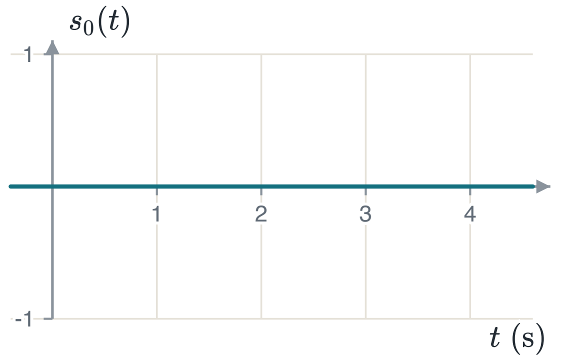
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

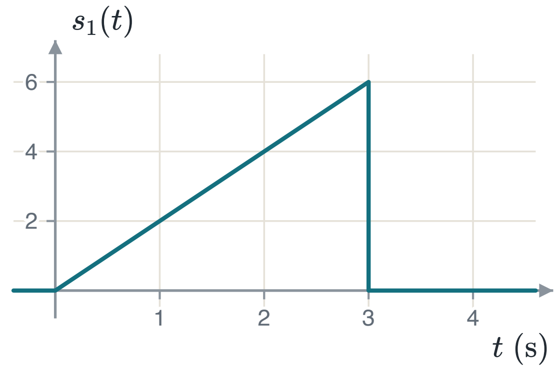
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-10 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 5.76$ W/Hz. Two equiprobable messages, $s_0(t)$ for “0” and $s_1(t)$ for “1”, are transmitted by the waveforms below. These signals are passed through the following matched-filter type demodulator. Here $\psi(t)$ is the unit-energy basis signal (pulse shape) that represents these signals. It is also used in the impulse response of the matched filter. The noise $w(t)$ is zero-mean white Gaussian noise, and $T = 4$ s. According to the information given above,





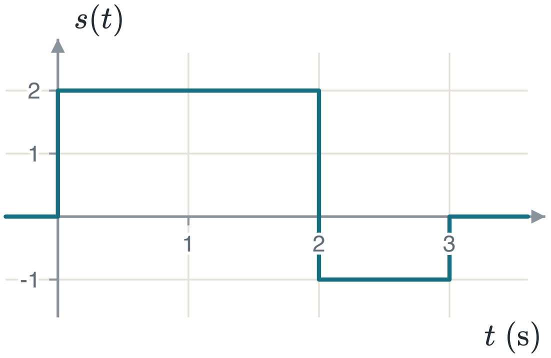
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

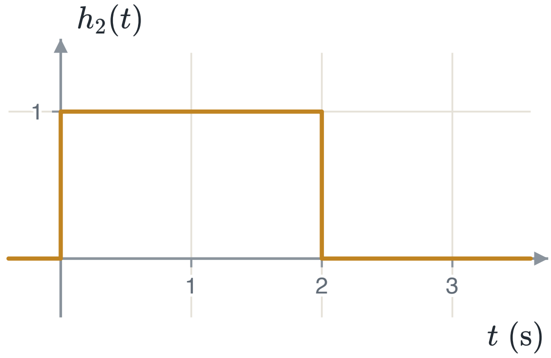
b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

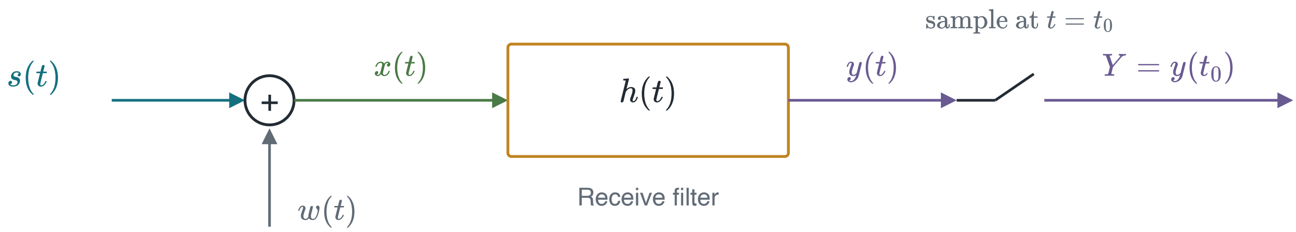
c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-11 · A FILTER THAT IS NOT MATCHED, OR A SAMPLE AT THE WRONG TIME

The pulse $s(t)$ below is received in additive white Gaussian noise $w(t)$ of two-sided power spectral density $N_0/2 = 0.5$ W/Hz, with $T = 3$ s. The received signal passes through a filter $h(t)$, and the output is sampled once, at $t = t_0$. Write $y_s(t)$ for the signal part of the filter output and $n(t)$ for its noise part. The output SNR at the sampling instant is $\eta = y_s^2(t_0)/E[n^2(t_0)]$. According to the information given above,







a) The filter is matched, $h(t) = s(T - t)$, and the output is sampled at $t_0 = T$. Find $y_s(T)$, $E[n^2(T)]$ and η.

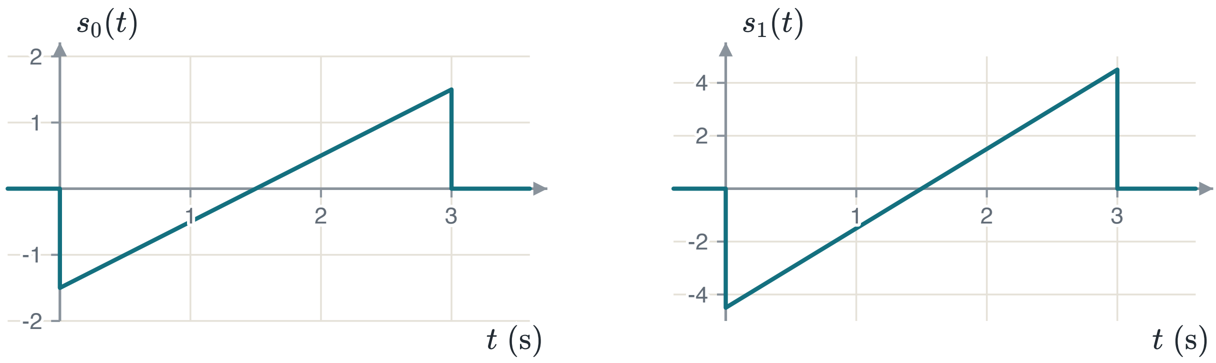
b) The clock of the same receiver runs early and samples at $t_0 = 2.5$ s. Find $y_s(2.5)$ and η.

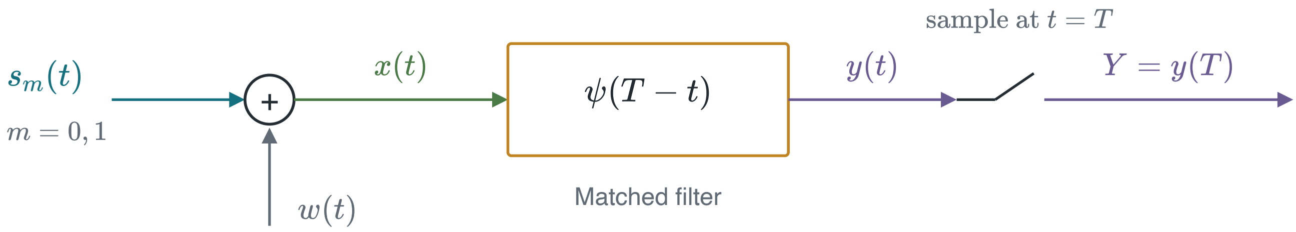
c) The matched filter is replaced by $h_2(t) = 1$ on $[0, 2]$, shown above. Find and plot the signal part $z_s(t)$ of its output. Give the best sampling instant and η there.

d) Give the loss of part (b) and the loss of part (c) against part (a) in dB. Which fault costs more?

D2-12 · MATCHED-FILTER DEMODULATOR FOR TWO DRAWN WAVEFORMS

Consider an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 1$ W/Hz. The message “0” is sent by $s_0(t)$ with probability 0.6, and “1” by $s_1(t)$ with probability 0.4. These signals are passed through the matched-filter type demodulator shown, with the unit-energy basis signal $\psi(t)$ and $T = 3$ s. According to the information given above,





a) Plot the impulse response of the matched filter, $h(t) = \psi(T-t)$.

b) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

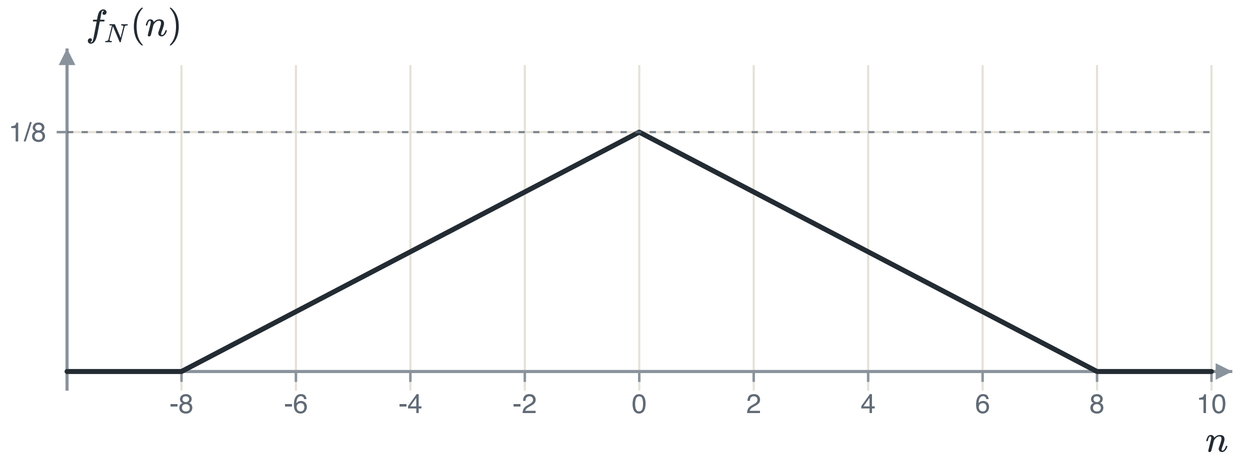
c) Find the optimal decision threshold (λ) and calculate the average probability of bit error (P_b). Use a table of the Q function for the numerical value.

D2-13 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is $Y = N$ if “0” is sent and $Y = 5 + N$ if “1” is sent. The probability density function of the noise sample N is shown below:

$$f_N(n) = \begin{cases} \frac{1}{8}\left(1 - \frac{|n|}{8}\right), & -8 \le n \le 8 \\ 0, & \text{otherwise.} \end{cases}$$

Assume that “0” and “1” are transmitted with equal probabilities.



a) Calculate the average probability of bit error (P_b) for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 3$, that is, decide “1” if $Y > 3$ and “0” if $Y < 3$.

b) Determine the optimal decision threshold (λ).

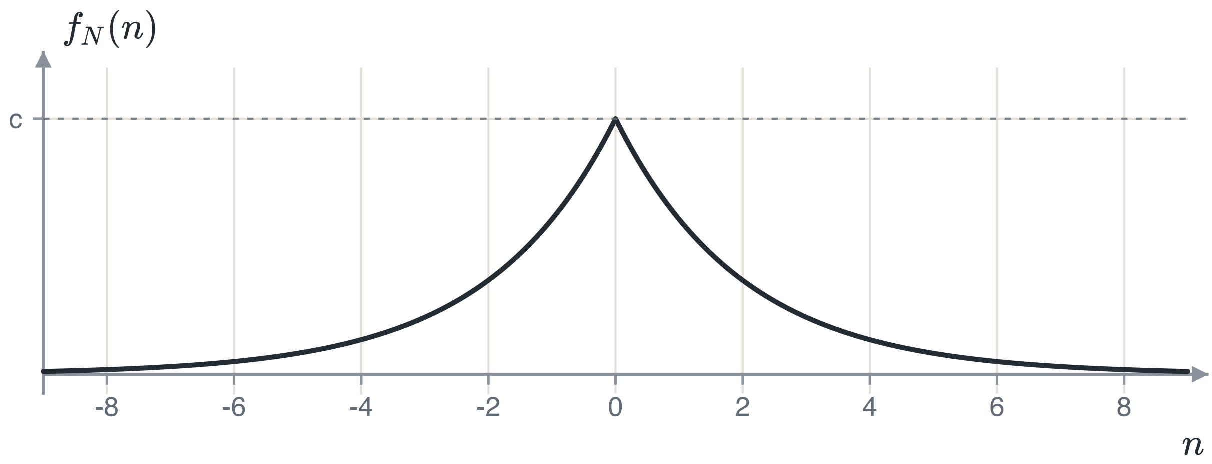
c) Calculate P_b for the λ value obtained in part (b).

D2-14 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is $Y = -2 + N$ if "0" is sent and $Y = 2 + N$ if "1" is sent. The PDF of the noise sample N is

$$f_N(n) = c\,e^{-|n|/2}, \qquad -\infty < n < \infty,$$

where c is a constant. Assume that "0" and "1" are transmitted with equal probabilities.



a) Find c. Then calculate the average probability of bit error for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 1$, that is, decide "1" if $Y > 1$ and "0" if $Y < 1$.

b) Determine the optimal decision threshold (λ).

c) Calculate the average probability of bit error (P_b) for the λ value obtained in part (b).

D2-15 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is denoted by Y. If "0" is transmitted, Y is exponentially distributed with mean 2. If "1" is transmitted, Y is exponentially distributed with mean 6. The input bits are equiprobable. An exponential PDF with mean μ is $f_Y(y) = \frac{1}{\mu}e^{-y/\mu}$ for $y \geq 0$, and 0 otherwise.

a) Calculate the average probability of bit error for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 4$, that is, decide "1" if $Y > 4$ and "0" if $Y < 4$.

b) Determine the optimal decision threshold (λ).

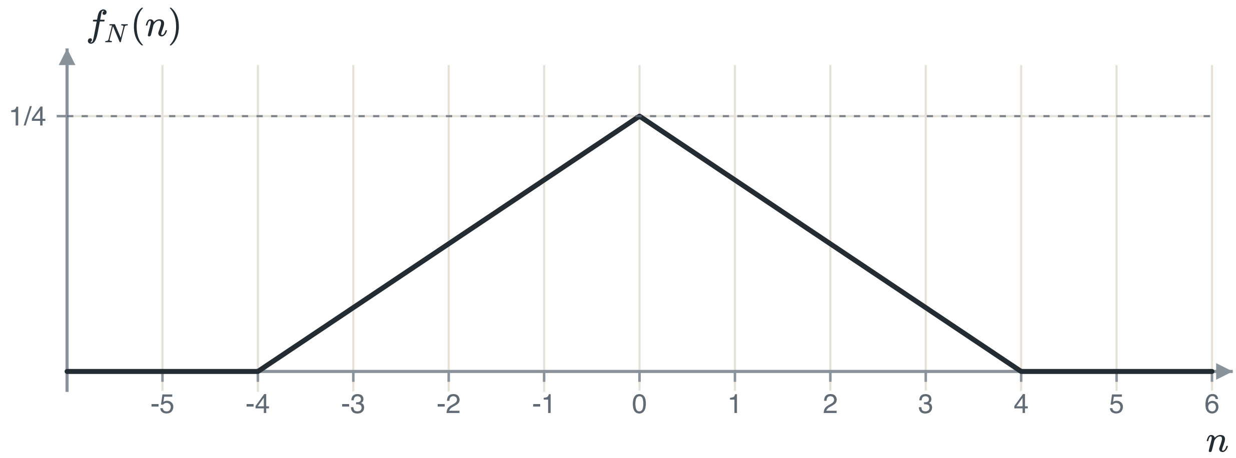
c) Calculate the average probability of bit error (P_b) for the λ value obtained in part (b).

D2-16 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is $Y = N$ if "0" is sent and $Y = 6 + N$ if "1" is sent. The noise sample N has the triangular PDF shown below:

$$f_N(n) = \begin{cases} \frac{1}{4}\left(1 - \frac{|n|}{4}\right), & -4 \le n \le 4 \\ 0, & \text{otherwise.} \end{cases}$$

Assume that "0" and "1" are transmitted with equal probabilities.



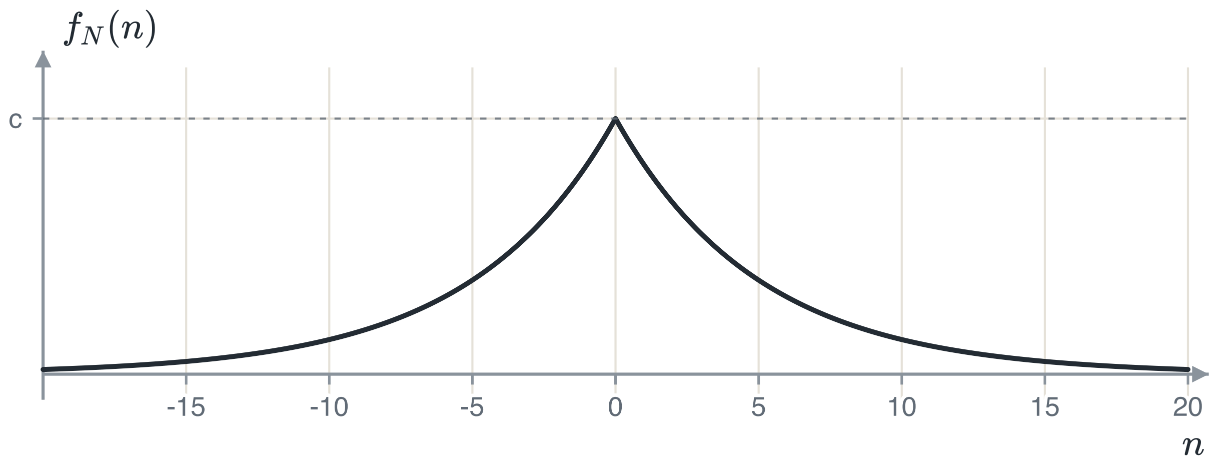
- **a)** Calculate the average probability of bit error for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 5$, that is, decide "1" if $Y > 5$ and "0" if $Y < 5$.
- **b)** Determine the optimal decision threshold (λ).
- **c)** Calculate the average probability of bit error (P_b) for the λ value obtained in part (b).

D2-17 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is $Y = N$ if "0" is sent and $Y = 8 + N$ if "1" is sent. The PDF of the noise sample N is

$$f_N(n) = c\,e^{-|n|/5}, \qquad -\infty < n < \infty,$$

where c is a constant. Assume that "0" and "1" are transmitted with equal probabilities.



a) Find c. Then calculate the average probability of bit error for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 3$, that is, decide "1" if $Y > 3$ and "0" if $Y < 3$.

b) Determine the optimal decision threshold (λ).

c) Calculate the average probability of bit error (P_b) for the λ value obtained in part (b).

D2-18 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is denoted by Y. If "0" is transmitted, Y is exponentially distributed with mean 3. If "1" is transmitted, Y is exponentially distributed with mean 12. The input bits are equiprobable. An exponential PDF with mean μ is $f_Y(y) = \frac{1}{\mu}e^{-y/\mu}$ for $y \geq 0$.

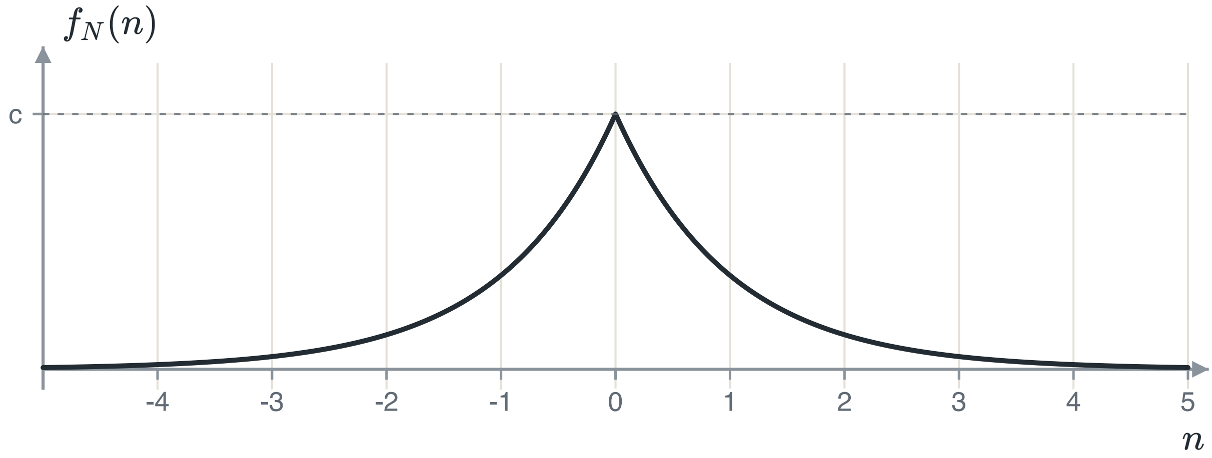
a) Calculate the average probability of bit error for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 6$, that is, decide "1" if $Y > 6$ and "0" if $Y < 6$.

b) Determine the optimal decision threshold (λ).

c) Calculate the average probability of bit error (P_b) for the λ value obtained in part (b).

D2-19 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is $Y = -3 + N$ if "0" is sent and $Y = 3 + N$ if "1" is sent. The PDF of the noise sample N is $f_N(n) = c\,e^{-|n|}$ for all n, where c is a constant. The bit "0" is sent with probability 0.8 and "1" with probability 0.2.



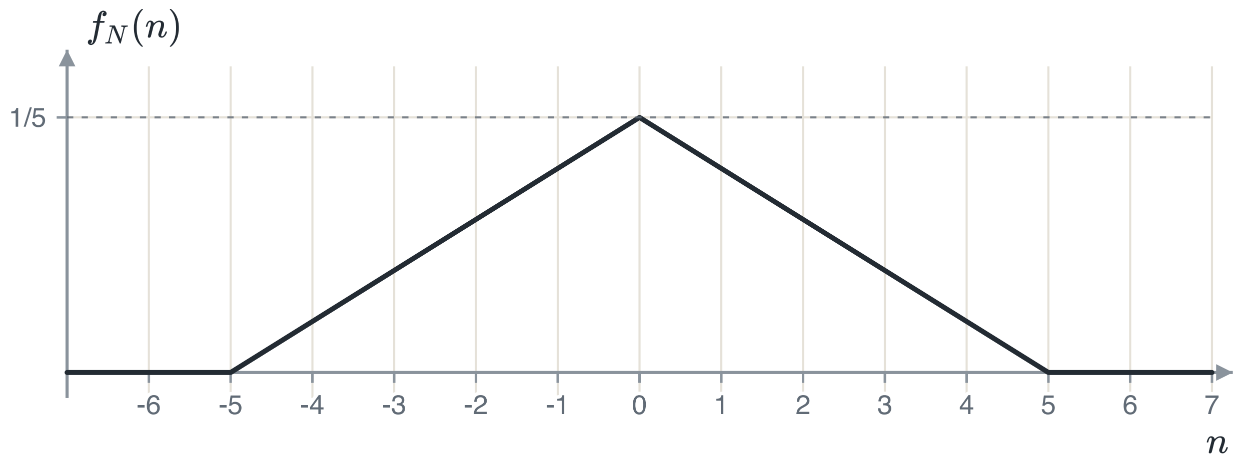
a) Find c. Then calculate P_b for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 0$.

b) Determine the optimal decision threshold (λ) for these priors.

c) Calculate P_b for the λ value obtained in part (b).

D2-20 · BINARY DECISION IN NOISE THAT IS NOT GAUSSIAN

In a binary digital communication system, the received sample is $Y = N$ if "0" is sent and $Y = 4 + N$ if "1" is sent. The noise N has the triangular PDF $f_N(n) = \frac{1}{5}\left(1 - \frac{|n|}{5}\right)$ for $|n| \leq 5$, and 0 otherwise. The bit "0" is sent with probability 0.6 and "1" with probability 0.4.



a) Calculate P_b for the decision rule $Y \underset{0}{\overset{1}{\gtrless}} 2$, the midpoint between the two signal values.

b) Determine the optimal decision threshold (λ) for these priors.

c) Calculate P_b for the λ value obtained in part (b).

D2-21 · BINARY DECISION ON A COUNT

An on-off optical link sends a light pulse for "1" and no pulse for "0". The receiver counts the photons that arrive in one bit interval. Background light gives a few counts even when no pulse is sent. The count Z is a Poisson random variable with mean $m_0 = 2$ if "0" is sent and $m_1 = 8$ if "1" is sent:

$$P(Z = k \mid m) = \frac{e^{-m} m^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

The count takes whole values only. According to this information,

a) Assume equal priors. Show that the optimal rule compares Z with a threshold, and find the smallest count that is decided as "1".

b) Calculate the conditional error probabilities $P(e \mid 0)$ and $P(e \mid 1)$ of this rule.

c) The bit "1" is now sent with probability 0.2. Find the optimal rule for these priors.

d) Calculate P_b for the rule of part (c). Compare it with P_b of the rule of part (a) at the same priors.

D2-22 · BINARY PAM WITH PRIORS, UNEQUAL VARIANCES OR FOLDED OUTPUTS

In a binary pulse amplitude modulation (PAM) communication system, “0” and “1” bits are transmitted. At the output of the matched filter, we observe $Y = |N_0|$ when “0” is transmitted and $Y = |N_1|$ when “1” is transmitted. Here N_0 and N_1 are independent Gaussian random variables with $N_0 \sim \mathcal{N}(\mu = 0, \sigma^2 = 1)$ and $N_1 \sim \mathcal{N}(\mu = 0, \sigma^2 = 4)$. Assume that “0” and “1” are transmitted with equal probabilities.

a) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

b) Find the optimal decision threshold (λ).

c) Calculate the average probability of bit error (P_b). Give the result as a numerical value, using a table of the Q function.

D2-23 · BINARY PAM WITH PRIORS, UNEQUAL VARIANCES OR FOLDED OUTPUTS

Assume that in a binary pulse amplitude modulation (PAM) communication system, “0” and “1” bits occur with probabilities 0.6 and 0.4. At the output of the matched filter, the output signal has a Gaussian distribution with mean 3 and variance 4 if “1” is transmitted. It has a Gaussian distribution with mean 0 and variance 1 if “0” is transmitted. According to this information,

a) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

b) Find the optimal decision rule. Show that it needs two thresholds, and give both.

c) Calculate the average probability of bit error (P_b) as a numerical value, using a table of the Q function.

D2-24 · BINARY PAM WITH PRIORS, UNEQUAL VARIANCES OR FOLDED OUTPUTS

Assume that in a binary pulse amplitude modulation (PAM) communication system, “0” and “1” bits occur with probabilities 0.7 and 0.3. Consider a signal detector with the input

$$Y = \begin{cases} 5 + N, & \text{if 1 is sent} \\ N, & \text{if 0 is sent} \end{cases}$$

where N is a Gaussian random variable with mean 0 and variance 4. According to this information,

a) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

b) Find the optimal decision threshold (λ).

c) Calculate the average probability of bit error (P_b) as a numerical value, using a table of the Q function.

D2-25 · BINARY PAM WITH PRIORS, UNEQUAL VARIANCES OR FOLDED OUTPUTS

In a binary pulse amplitude modulation (PAM) communication system, the matched-filter output is $Y = -2 + N$ when "0" is sent and $Y = 2 + N$ when "1" is sent. The noise N is Gaussian with mean 0 and variance 1. The bit "1" occurs three times as often as the bit "0". According to this information,

a) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

b) Find the optimal decision threshold (λ).

c) Calculate P_b as a numerical value, using a table of the Q function.

d) A receiver ignores the priors and uses $\lambda = 0$. Calculate its P_b and compare.

D2-26 · A NYQUIST PULSE THAT IS NOT A RAISED COSINE

A binary baseband link uses the pulse

$$s(t) = \mathrm{sinc}(at)\,\mathrm{sinc}(bt), \qquad a \geq b > 0,$$

where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. The channel is an ideal lowpass channel that passes the band $|f| \leq B = 2.5$ kHz. Use the transform pair $\mathrm{sinc}(at) \leftrightarrow \frac{1}{a}\Pi(f/a)$, where $\Pi(f/a) = 1$ for $|f| < a/2$ and 0 otherwise. According to this information,

a) Find the spectrum $S(f)$ and sketch it. Give $S(0)$, the frequency where its flat top ends and the frequency where it reaches zero.

b) Choose a and b so that the pulse has zero intersymbol interference at $R_b = 4$ kb/s and exactly fills the channel.

c) With these a and b, is the pulse free of intersymbol interference at $R_b = 2$ kb/s? At $R_b = 5$ kb/s? Give a reason for each.

d) Find the roll-off factor $\alpha = (B - W)/W$ at $R_b = 4$ kb/s, where $W = R_b/2$. Then redesign a and b for $R_b = 4.5$ kb/s in the same channel, and give the new α.

D2-27 · WORKING BACK FROM A TARGET ERROR OR THRESHOLD

In a binary pulse amplitude modulation (PAM) communication system, the matched-filter output is $Y = N$ when "0" is sent and $Y = 3 + N$ when "1" is sent. The noise N is Gaussian with mean 0 and variance 1. A designer wants the optimal threshold to sit at $\lambda = 2$. According to this information,

a) Write $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$. Find the optimal threshold and P_b for equal priors.

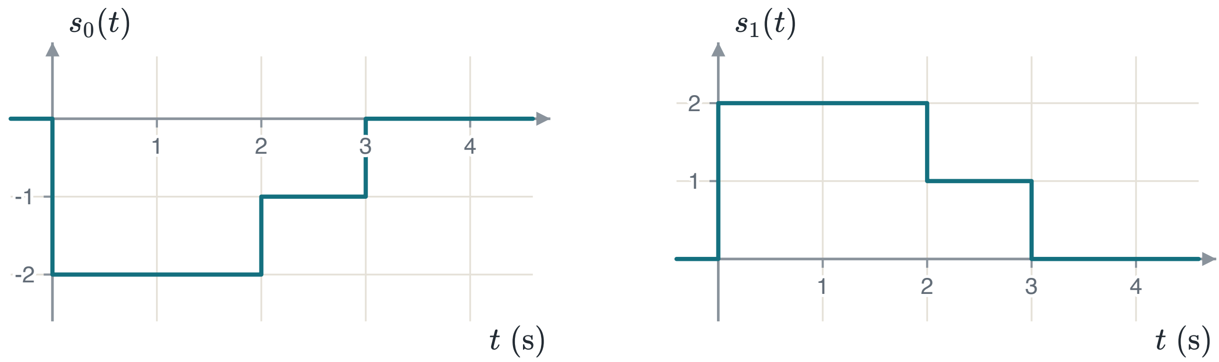
b) Find the probability p_0 of "0" that makes $\lambda = 2$ the optimal threshold.

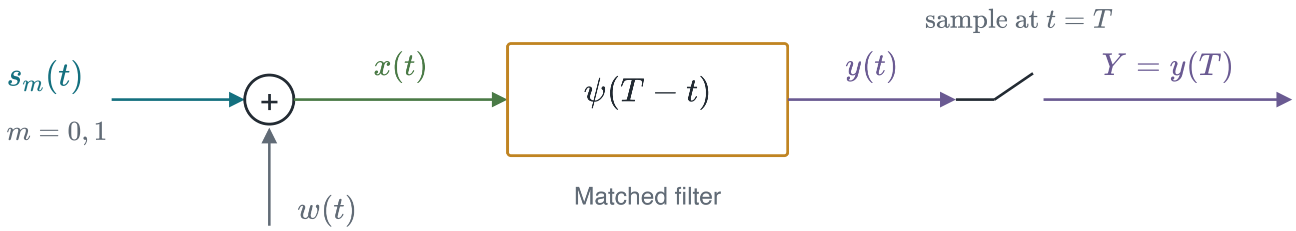
c) Calculate P_b for this prior with $\lambda = 2$.

d) With this prior, the receiver keeps the threshold of part (a). Calculate its P_b and compare.

D2-28 · WORKING BACK FROM A TARGET ERROR OR THRESHOLD

Two equiprobable messages, $s_0(t)$ for "0" and $s_1(t)$ for "1", are transmitted by the waveforms below, with $T = 4$ s. They pass through the matched-filter type demodulator shown, with the unit-energy basis signal $\psi(t)$. The channel adds zero-mean white Gaussian noise of two-sided power spectral density $N_0/2$, which is not yet known. The link must reach $P_b \le 10^{-3}$. According to the information given above,





a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

b) Write the optimal threshold and P_b as a function of $N_0/2$.

c) Find the largest $N_0/2$ that meets $P_b \le 10^{-3}$. Use $Q(3.09) = 1.00 \times 10^{-3}$.

d) The message "0" is now sent as $s_0(t) = 0$, with the same $s_1(t)$. Find the largest $N_0/2$ again.

D2-29 · DETECTION WITH A BANDWIDTH OR WITH INTERFERENCE

A binary antipodal baseband link sends $R_b = 12$ kb/s with raised-cosine pulses of roll-off factor $\alpha = 0.25$. The matched-filter sample is $Y = \sqrt{E_b} + N$ for "1" and $Y = -\sqrt{E_b} + N$ for "0", and the bits are equiprobable. The received power is $P = 3.6\ \mu$W. N is Gaussian with mean 0 and variance $N_0/2$, where $N_0 = 5 \times 10^{-11}$ W/Hz. According to this information,

a) Find the Nyquist bandwidth W, the frequency f_1 where the roll-off starts, and the transmission bandwidth B_T.

b) Find E_b. Write $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$, and give the optimal threshold.

c) Calculate P_b with a table of the Q function.

d) The channel offers $B_T = 9$ kHz. The rate is raised to fill it, with the same α and the same received power. Find the new R_b and P_b.

D2-30 · DETECTION WITH A BANDWIDTH OR WITH INTERFERENCE

In a binary PAM system, the pulse of each bit spreads into the next bit interval. The matched-filter sample of the current bit is $Y = a + c + N$. Here $a = 1.8$ for "1" and $a = -1.8$ for "0". The term c comes from the previous bit: $c = 0.6$ if that bit was "1" and $c = -0.6$ if it was "0". All bits are independent and equiprobable, and N is Gaussian with mean 0 and variance 0.36. According to this information,

a) Determine the conditional PDFs, $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$, averaged over the previous bit.

b) Find the optimal decision threshold (λ).

c) Calculate P_b with a table of the Q function. Compare it with P_b when the interference is removed ($c = 0$).

MODULE

3

Geometric Representation of Signals

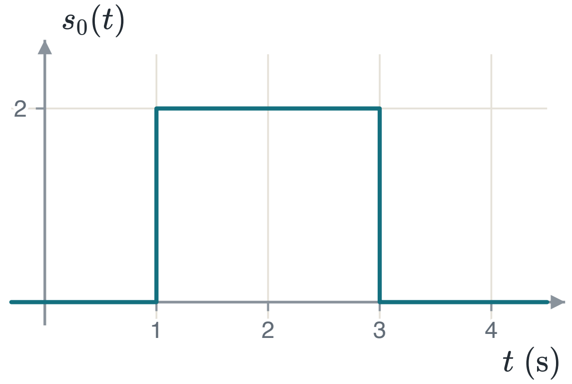
30 questions on geometric representation of signals. Write your reasoning in the space under each one.

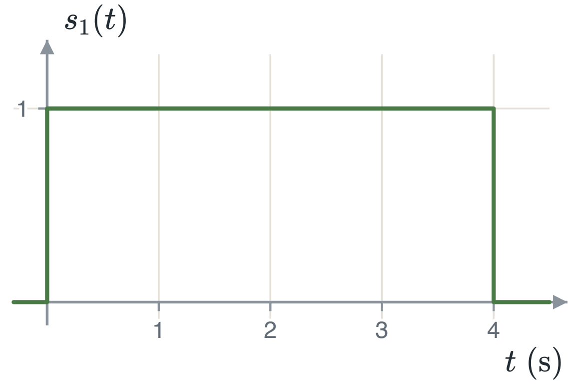
D3–01 · DRAWN WAVEFORMS AND GRAM–SCHMIDT

In an additive white Gaussian noise channel, two equiprobable messages are transmitted by the waveforms $s_0(t)$ and $s_1(t)$ below, with $T = 4$ s. Before a receiver is designed, the pair is written in geometric form. The correlation coefficient of the pair is

$$\rho_{01} = \frac{1}{\sqrt{E_0 E_1}} \int_0^T s_0(t)\, s_1(t)\, dt.$$

According to the information given above,





a) Find an orthonormal basis for $\{s_0(t), s_1(t)\}$ by the Gram–Schmidt procedure, starting with $s_0(t)$. Plot the basis functions.

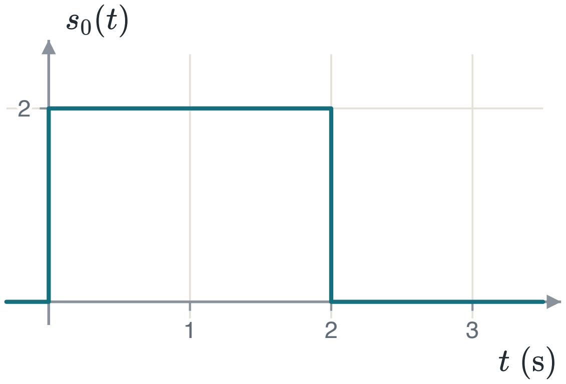
b) Find the signal vectors $\mathbf{s}_0$, $\mathbf{s}_1$ and the energies E_0, E_1.

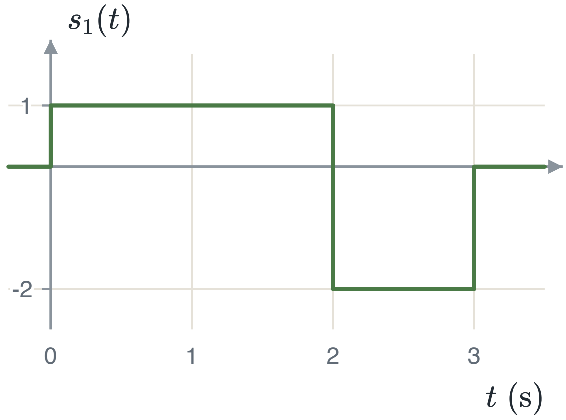
c) Calculate the distance $d_{01} = \|\mathbf{s}_0 - \mathbf{s}_1\|$ and the correlation coefficient ρ_{01}.

d) Draw the signal constellation. How many dimensions does the signal set need?

D3-02 · DRAWN WAVEFORMS AND GRAM-SCHMIDT

Two equiprobable messages are transmitted by the waveforms $s_0(t)$ and $s_1(t)$ shown below, with $T = 3$ s. The correlation coefficient of the pair is $\rho_{01} = \langle s_0, s_1 \rangle / \sqrt{E_0 E_1}$, where $\langle s_0, s_1 \rangle = \int_0^T s_0(t) s_1(t)\, dt$. According to the information given above,





a) Use the Gram–Schmidt procedure, starting with $s_0(t)$, to find an orthonormal basis. Plot the basis functions.

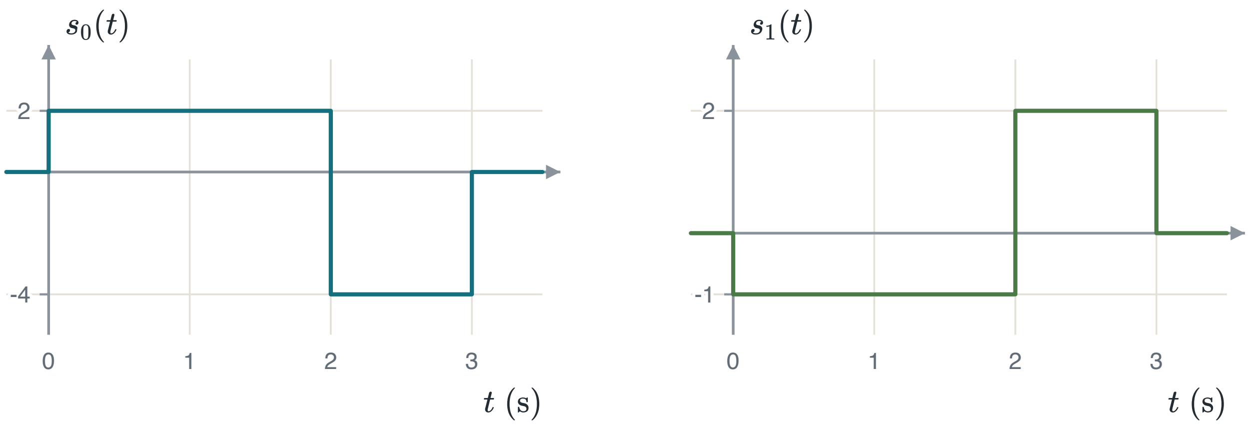
b) Find the signal vectors and the energies E_0 and E_1.

c) Calculate d_{01} and ρ_{01}.

d) Draw the signal constellation and calculate the average symbol energy $E_{s,av}$.

D3-03 · DRAWN WAVEFORMS AND GRAM–SCHMIDT

In an additive white Gaussian noise channel, two equiprobable messages are transmitted by the waveforms below, with $T = 3$ s. According to the information given above,



a) Apply the Gram–Schmidt procedure, starting with $s_0(t)$. Show that it stops after one basis function, and plot that function.

b) Find the coordinates of the two signals and their energies.

c) Calculate the distance d_{01} and the correlation coefficient $\rho_{01} = \langle s_0, s_1 \rangle / \sqrt{E_0 E_1}$.

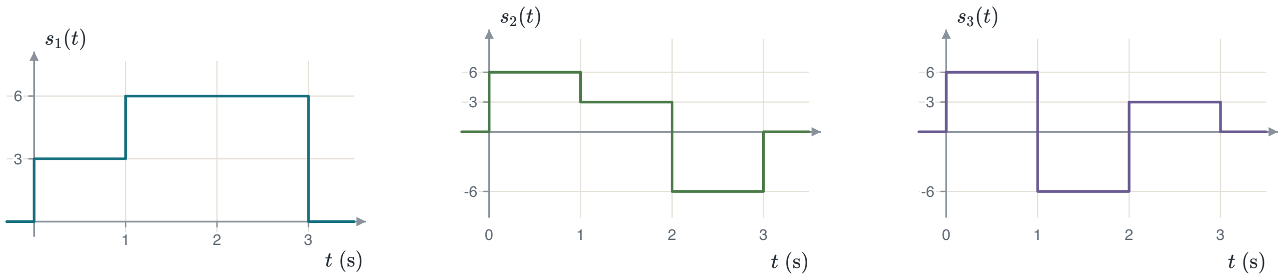
d) Draw the signal constellation and give $E_{s,av}$.

D3-04 · REMOVING THE AVERAGE POINT

Three equiprobable messages are sent with the waveforms $s_1(t)$, $s_2(t)$ and $s_3(t)$ shown below, with $T = 3$ s. A new set is built by subtracting their average waveform from each of them:

$$\bar{s}(t) = \frac{1}{3}\left[s_1(t) + s_2(t) + s_3(t)\right], \qquad u_k(t) = s_k(t) - \bar{s}(t), \quad k = 1, 2, 3.$$

The set $\{u_1, u_2, u_3\}$ is called a simplex set. According to the information given above,



a) Show that s_1, s_2 and s_3 are orthogonal and have one common energy E_s. Give an orthonormal basis and the three signal vectors.

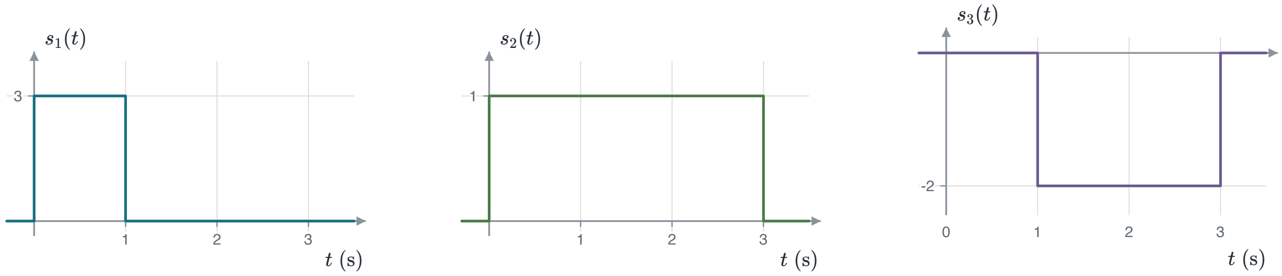
b) Find and sketch $\bar{s}(t)$ and the three simplex waveforms $u_k(t)$. Give their signal vectors in the same basis.

c) Calculate the energy E_u of each u_k, the inner products $\langle u_j, u_k \rangle$ and the correlation coefficient $\rho = \langle u_j, u_k \rangle / E_u$. Compare the distances in the two sets.

d) Now start from M orthogonal waveforms of energy E_s. Show that $E_u = (1 - 1/M)E_s$ and $\rho = -1/(M-1)$. Give the energy saving $10\log_{10}(E_s/E_u)$ in dB for $M = 2, 3, 4, 8, 16$.

D3-05 · DRAWN WAVEFORMS AND GRAM–SCHMIDT

Three equiprobable messages are transmitted by the waveforms $s_1(t)$, $s_2(t)$ and $s_3(t)$ shown below, with $T = 3$ s. According to the information given above,



a) Find an orthonormal basis by the Gram–Schmidt procedure, taking the signals in the order s_1, s_2, s_3. Plot the basis functions.

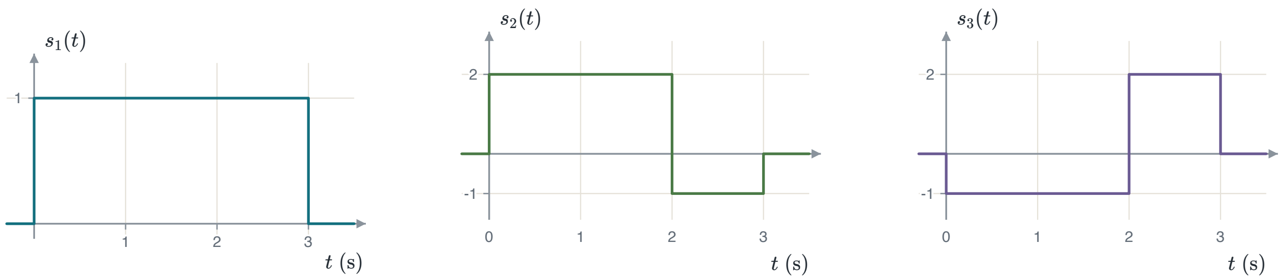
b) Find the three signal vectors and their energies.

c) Calculate the three distances between the signals and give d_{min}.

d) Draw the signal constellation and state the number of dimensions.

D3-06 · DRAWN WAVEFORMS AND GRAM–SCHMIDT

Three equiprobable symbols are represented by the waveforms shown below on $0 \le t \le T$, with $T = 3$ s. According to the information given above,



a) Use the Gram–Schmidt procedure in the order s_1, s_2, s_3 to find an orthonormal basis. Plot the basis functions.

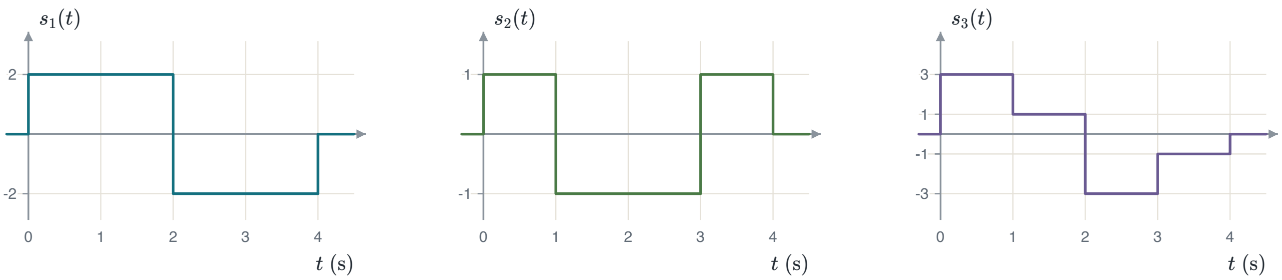
b) Find the signal vectors.

c) Calculate $E_{s,av}$ and the minimum distance d_{min}.

d) Draw the signal constellation, and write $s_3(t)$ as a combination of $s_1(t)$ and $s_2(t)$.

D3-07 · DRAWN WAVEFORMS AND GRAM-SCHMIDT

Three equiprobable messages are transmitted by the waveforms shown below, with $T = 4$ s. According to the information given above,



a) Show that $s_1(t)$ and $s_2(t)$ are orthogonal, and give their correlation coefficient $\rho_{12} = \langle s_1, s_2 \rangle / \sqrt{E_1 E_2}$.

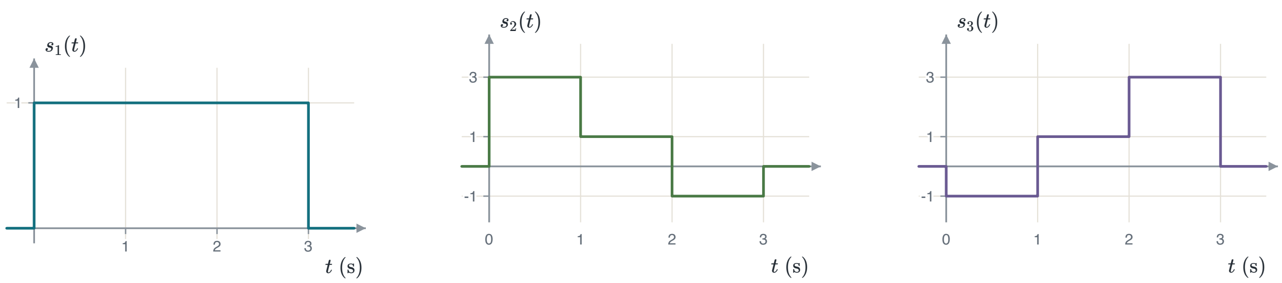
b) Find an orthonormal basis for all three signals by the Gram–Schmidt procedure in the order s_1, s_2, s_3, and the three signal vectors.

c) Calculate the three distances and $d_{\min}$. Which pair is closest?

d) Draw the signal constellation and calculate $E_{s,av}$.

D3-08 · DRAWN WAVEFORMS AND GRAM-SCHMIDT

Three equiprobable symbols are transmitted with the staircase waveforms shown below on $0 \le t \le T$, with $T = 3$ s. According to the information given above,



a) Find an orthonormal basis by the Gram–Schmidt procedure in the order s_1, s_2, s_3. Plot the basis functions.

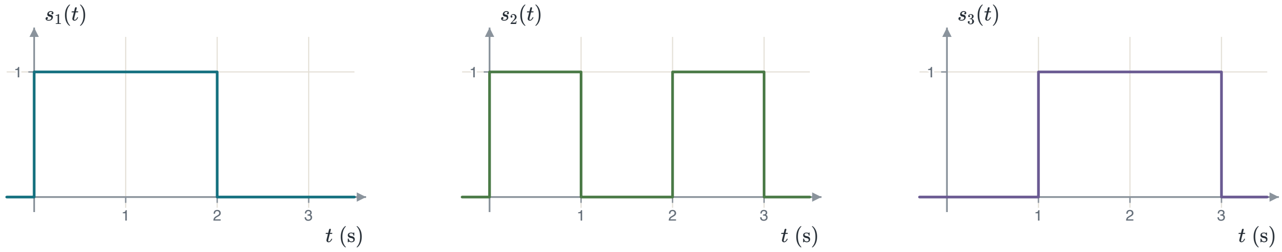
b) Find the signal vectors and the symbol energies.

c) Calculate all distances, $d_{\min}$, and the number of nearest neighbours of each symbol.

d) Draw the constellation. The three points lie on one straight line. Explain why the set still needs two dimensions.

D3-09 · DRAWN WAVEFORMS AND GRAM–SCHMIDT

Three equiprobable messages are transmitted by the waveforms shown below, with $T = 3$ s. Each waveform is 1 on two of the three unit intervals and 0 on the third. According to the information given above,



a) Apply the Gram–Schmidt procedure in the order s_1, s_2, s_3 and find an orthonormal basis. Plot the basis functions.

b) Find the three signal vectors.

c) Calculate the energies and the three distances.

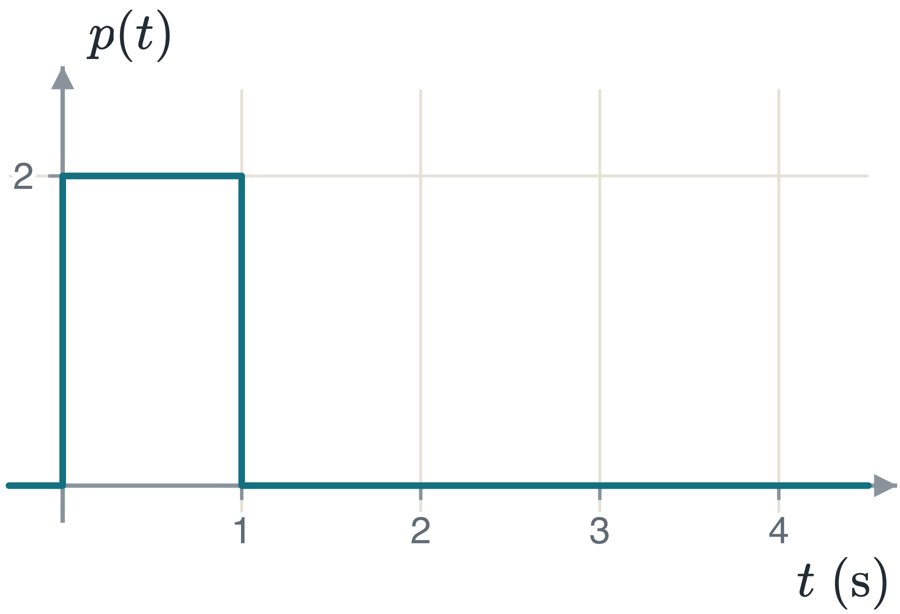
d) How many dimensions does the set need, and what shape do the three points form?

D3-10 · COMPARING TWO DESCRIPTIONS OR TWO DESIGNS

Let $p(t)$ be the rectangular pulse of height 2 on $0 \le t < 1$ s, zero elsewhere. Two sets of four equiprobable waveforms on $0 \le t \le 4$ s are built from its shifts:

$$\begin{aligned} &\text{Set A:} \quad a_i(t) = p(t-i), \quad i = 0,1,2,3, \\ &\text{Set B:} \quad b_0(t) = p(t) + p(t-1), \quad b_1(t) = p(t-2) + p(t-3), \\ &\qquad\qquad b_2(t) = p(t) + p(t-2), \quad b_3(t) = p(t-1) + p(t-3). \end{aligned}$$

Each symbol carries $\log_2 4 = 2$ bits, so the energy per bit is $E_b = E_{s,av}/2$. According to the information given above,



a) Show that $\psi_i(t) = \frac{1}{2}p(t-i)$, $i = 0,1,2,3$, is an orthonormal basis for both sets. Find the signal vectors of set A and of set B.

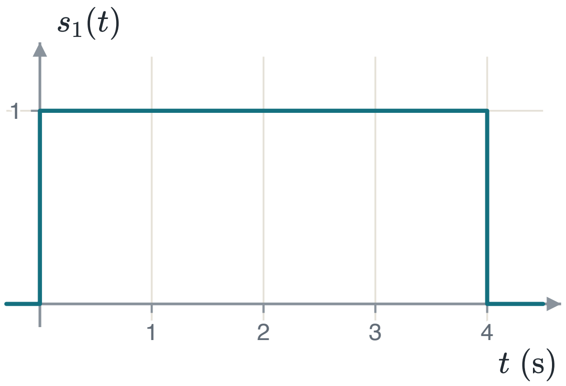
b) Calculate $E_{s,av}$ and E_b for each set.

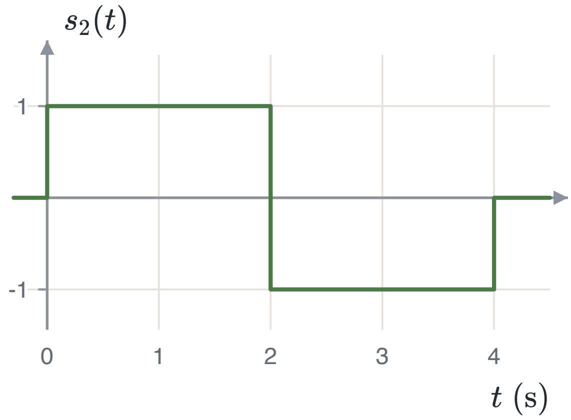
c) Calculate the six distances within set B. Give $d_{\min}$ and the number of nearest neighbours of each symbol, for both sets.

d) Compare the sets by $d^2_{\min}/E_b$. Which set is better, and by how many dB? Where does the other set spend its extra energy?

D3-11 · DRAWN WAVEFORMS AND GRAM–SCHMIDT

Four equiprobable symbols are transmitted by the waveforms $s_1(t)$ and $s_2(t)$ shown below and by their negatives, $s_3(t) = -s_1(t)$ and $s_4(t) = -s_2(t)$, with $T = 4$ s. According to the information given above,





a) Find an orthonormal basis for the four signals by the Gram–Schmidt procedure, in the order s_1, s_2, s_3, s_4.

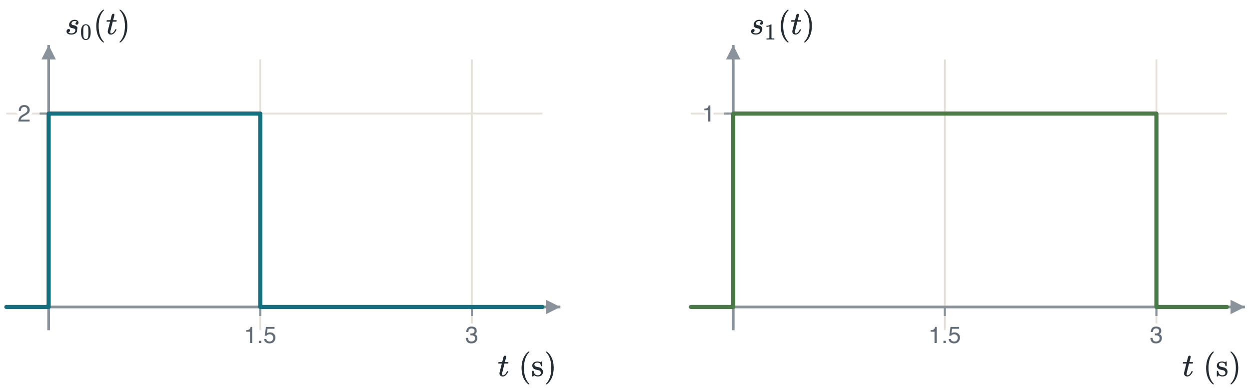
b) Find the four signal vectors.

c) Calculate $E_{s,av}$, the minimum distance $d_{\min}$, and the number of nearest neighbours of each symbol.

d) Draw the signal constellation and give the ratio $d_{\min}^2/E_{s,av}$.

D3-12 · DRAWN WAVEFORMS AND GRAM-SCHMIDT

Two equiprobable messages are transmitted by the waveforms $s_0(t)$ and $s_1(t)$ shown below, with $T = 3$ s. Note that $s_0(t)$ changes value at $t = 1.5$ s. According to the information given above,



a) Find an orthonormal basis by the Gram–Schmidt procedure, starting with $s_0(t)$. Plot the basis functions.

b) Find the signal vectors and the energies.

c) Calculate d_{01} and $\rho_{01} = \langle s_0, s_1 \rangle / \sqrt{E_0 E_1}$.

d) Draw the signal constellation and calculate $d_{01}^2 / E_{s,av}$.

D3-13 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = \sqrt{18} \cos\left(4000\pi t + \frac{\pi(2k-1)}{6}\right), \qquad k \in \{1, \ldots, 6\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the signal vectors and draw the signal constellation.

c) Calculate the symbol energies and the average symbol energy $E_{s,av}$.

d) Determine the minimum distance $d_{\min}$ and the number of nearest neighbours of each symbol.

D3–14 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$\begin{aligned} s_1(t) &= 0, \\ s_2(t) &= 3\sqrt{2}\cos(6000\pi t), \\ s_3(t) &= 3\sqrt{2}\cos\left(6000\pi t + \frac{\pi}{2}\right), \\ s_4(t) &= 6\cos\left(6000\pi t + \frac{\pi}{4}\right), \end{aligned} \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the four signal vectors and draw the signal constellation.

c) Calculate the symbol energies and $E_{s,av}$.

d) Determine $d_{\min}$, the number of nearest neighbours of each symbol, and $d_{\min}^2/E_{s,av}$.

D3–15 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = (2k-5)\sqrt{2}\cos(5000\pi t), \qquad k \in \{1,2,3,4\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Show that one basis function is enough for this set. Give it and show that it has unit energy.

b) Find the signal coordinates and draw the signal constellation.

c) Calculate the symbol energies and $E_{s,av}$.

d) Determine $d_{\min}$, the number of nearest neighbours of each symbol, and their average.

D3–16 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = 2\cos\left(2000\pi t + \frac{k\pi}{4}\right), \qquad k \in \{0, 1, \ldots, 7\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the signal vectors and draw the signal constellation.

c) Calculate $E_{s,av}$ and the minimum distance $d_{\min}$.

d) Give the number of nearest neighbours of each symbol and the ratio $d_{\min}^2 / E_{s,av}$.

D3–17 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = \sqrt{10}\cos\left(6000\pi t + \frac{(2k-1)\pi}{4}\right), \qquad k \in \{1, 2, 3, 4\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the signal vectors and draw the signal constellation.

c) Calculate $E_{s,av}$.

d) Determine $d_{\min}$, the number of nearest neighbours, and the largest distance between two symbols.

D3-18 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_{m,n}(t) = \sqrt{2}\left[(2m-5)\cos(4000\pi t) - (2n-5)\sin(4000\pi t)\right],$$
$$m, n \in \{1,2,3,4\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the signal vector of $s_{m,n}$ and draw the signal constellation.

c) Calculate the symbol energies that occur and $E_{s,av}$.

d) Determine $d_{\min}$ and the number of nearest neighbours of a corner, an edge and an inner symbol. Give the average.

D3-19 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = \begin{cases} 2\sqrt{2}\cos\left(2000\pi t + \dfrac{(k-1)\pi}{2}\right), & k \in \{1,2,3,4\}, \\ 4\cos\left(2000\pi t + \dfrac{(2k-9)\pi}{4}\right), & k \in \{5,6,7,8\}, \end{cases} \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the signal vectors and draw the signal constellation.

c) Calculate the symbol energies and the average symbol energy $E_{s,av}$.

d) Determine the minimum distance $d_{\min}$ and the number of nearest neighbours of each symbol.

D3-20 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_1(t) = 0, \qquad s_k(t) = 2\sqrt{2}\cos\left(2000\pi t + \frac{2\pi(k-2)}{3}\right), \quad k \in \{2,3,4\}, \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the four signal vectors and draw the signal constellation.

c) Calculate the symbol energies and $E_{s,av}$.

d) Determine $d_{\min}$, the number of nearest neighbours of each symbol, and $d_{\min}^2/E_{s,av}$.

D3-21 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$\begin{aligned} s_1(t) &= 3\sqrt{2}\cos(2000\pi t), & s_3(t) &= -s_1(t), \\ s_2(t) &= 3\sqrt{2}\cos(3000\pi t), & s_4(t) &= -s_2(t), \end{aligned} \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Show that $\psi_1(t) = \sqrt{2}\cos(2000\pi t)$ and $\psi_2(t) = \sqrt{2}\cos(3000\pi t)$ form an orthonormal basis for the set.

b) Find the signal vectors and draw the signal constellation.

c) Calculate the symbol energies and $E_{s,av}$.

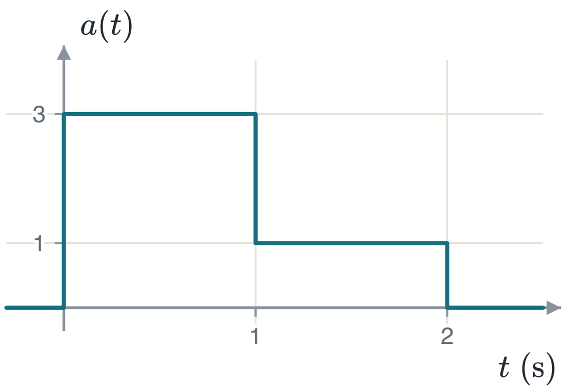
d) Determine $d_{\min}$ and the number of nearest neighbours of each symbol.

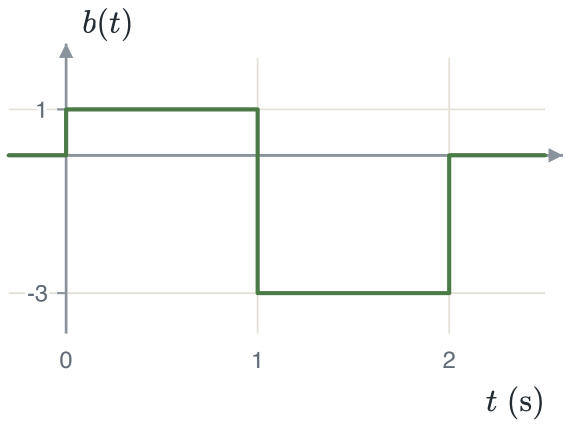
D3-22 · CARRIER WAVEFORMS BUILT FROM TWO PULSES

The baseband pulses $a(t)$ and $b(t)$ below, on $0 \le t \le 2$ s, build four carrier waveforms at $f_c = 1000$ Hz:

$$\begin{aligned} s_1(t) &= a(t)\cos(2\pi f_c t) - b(t)\sin(2\pi f_c t), \\ s_2(t) &= b(t)\cos(2\pi f_c t) - a(t)\sin(2\pi f_c t), \\ s_3(t) &= b(t)\cos(2\pi f_c t) + a(t)\sin(2\pi f_c t), \\ s_4(t) &= a(t)\cos(2\pi f_c t) + b(t)\sin(2\pi f_c t), \end{aligned} \qquad 0 \le t \le 2.$$

Each unit interval holds a whole number of carrier cycles. According to the information given above,





a) Find the energies E_a, E_b and the inner product $\langle a, b\rangle$. Show that on any unit interval $[k, k+1)$ the integrals of $\cos^2(2\pi f_c t)$ and $\sin^2(2\pi f_c t)$ are $\frac{1}{2}$, and the integral of $\cos(2\pi f_c t)\sin(2\pi f_c t)$ is 0.

b) Write the energy of each $s_k(t)$ and the six inner products $\langle s_j, s_k\rangle$ in terms of E_a, E_b and $\langle a, b\rangle$, and evaluate them. Can the set serve for 4-ary orthogonal signalling with coherent detection?

c) Find an orthonormal basis of four functions and the four signal vectors. Give the distance between two symbols and $d^2_{\min}/E_{s,av}$.

d) The pulse $b(t)$ is replaced by $2b(t)$. Which pairs stay orthogonal? Give the inner product and the correlation coefficient of each other pair.

D3-23 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_{m,n}(t) = \sqrt{2}\left[(2m-5)\cos(2000\pi t) - (2n-3)\sin(2000\pi t)\right],$$
$$m \in \{1,2,3,4\},\ n \in \{1,2\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

b) Find the signal vectors and draw the signal constellation.

c) Calculate the symbol energies and $E_{s,av}$.

d) Determine $d_{\min}$ and the number of nearest neighbours of each symbol. Give the average.

D3-24 · A CARRIER WAVEFORM FAMILY AND ITS CONSTELLATION

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = k\sqrt{2}\cos\left(3000\pi t + \frac{k\pi}{2}\right), \qquad k \in \{1,2,3,4\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel. According to the information given above,

a) Find an orthonormal basis for the signal set and show that it is orthonormal.

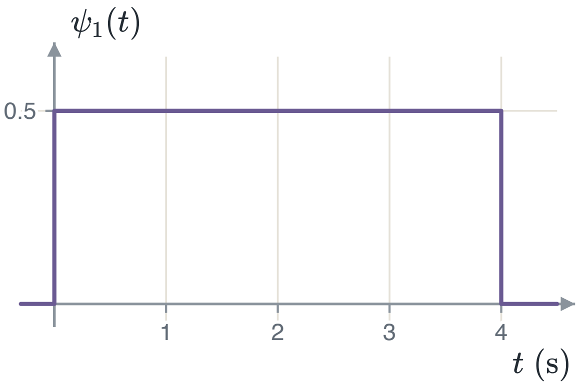
b) Find the four signal vectors and draw the signal constellation.

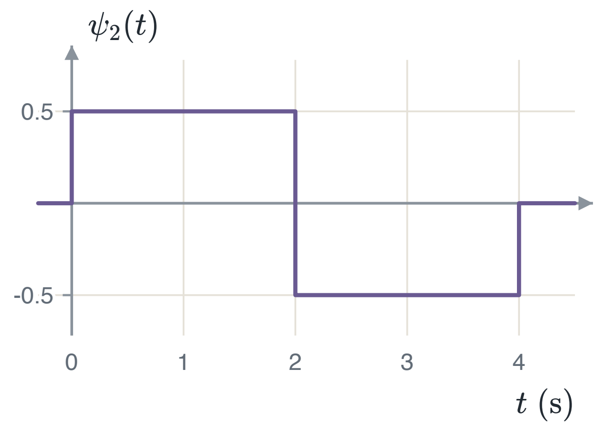
c) Calculate the symbol energies and $E_{s,av}$.

d) Calculate all six distances, $d_{\min}$, and the number of nearest neighbours of each symbol.

D3-25 · FROM A CONSTELLATION BACK TO THE WAVEFORMS

A signal set is built on the two basis functions $\psi_1(t)$ and $\psi_2(t)$ shown below, with $T = 4$ s. The four equiprobable symbols have the signal vectors $\mathbf{s}_1 = (2, 2)$, $\mathbf{s}_2 = (2, -2)$, $\mathbf{s}_3 = (-2, -2)$ and $\mathbf{s}_4 = (-2, 2)$. According to the information given above,





a) Show that $\psi_1(t)$ and $\psi_2(t)$ are orthonormal.

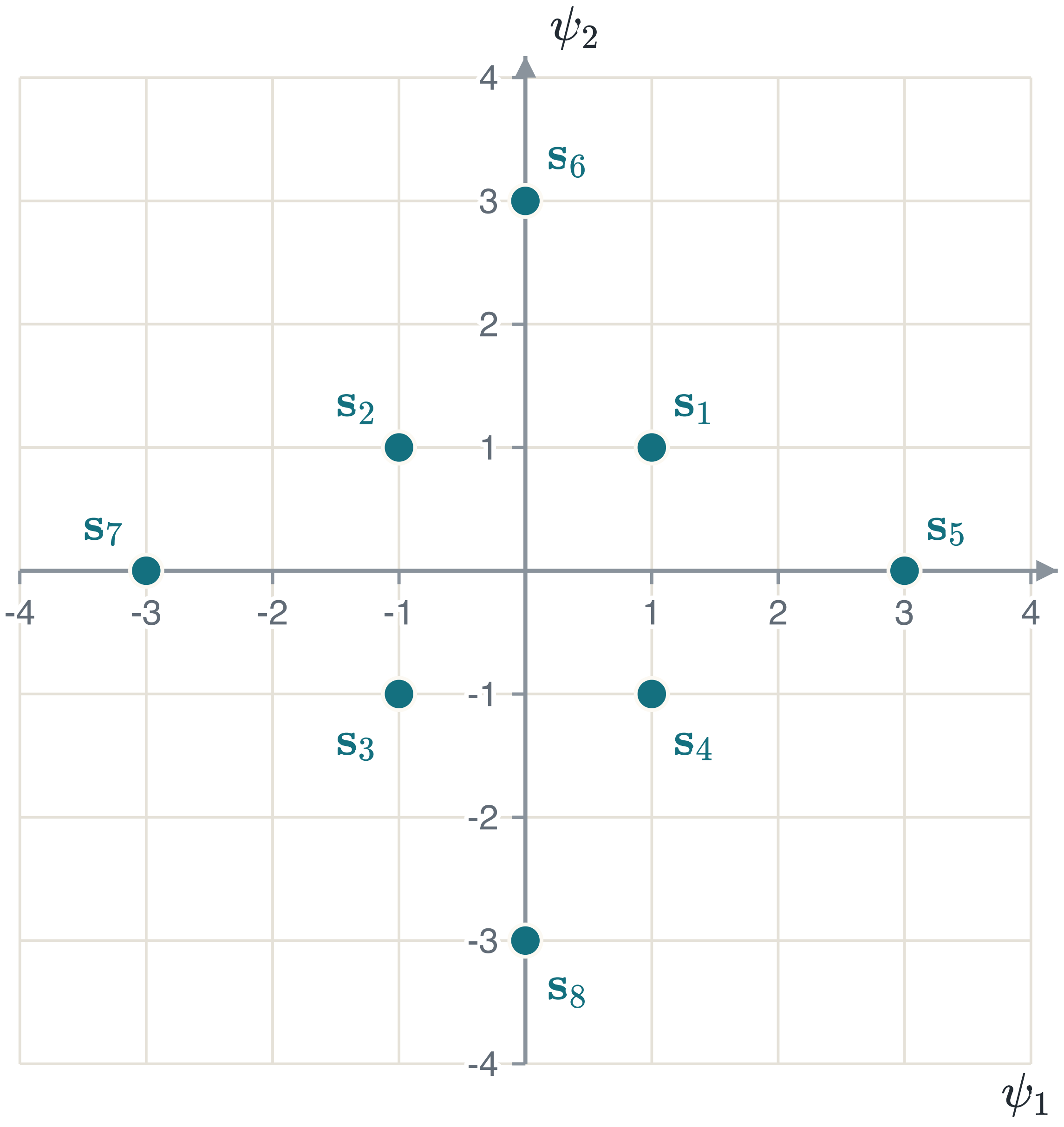
b) Write the four waveforms $s_1(t), \ldots, s_4(t)$ and sketch them.

c) Calculate the energy of each waveform from its sketch, and $E_{s,av}$.

d) Determine $d_{\min}$ and the number of nearest neighbours. Which pairs of waveforms are orthogonal?

D3-26 · FROM A CONSTELLATION BACK TO THE WAVEFORMS

An M-ary modulation scheme uses the basis $\psi_1(t) = \sqrt{2}\cos(4000\pi t)$ and $\psi_2(t) = -\sqrt{2}\sin(4000\pi t)$ on $0 \le t \le 1$, which is orthonormal. Its eight equiprobable symbols form the constellation below. According to the information given above,



a) Write each waveform in the form $s_k(t) = A_k \cos(4000\pi t + \theta_k)$.

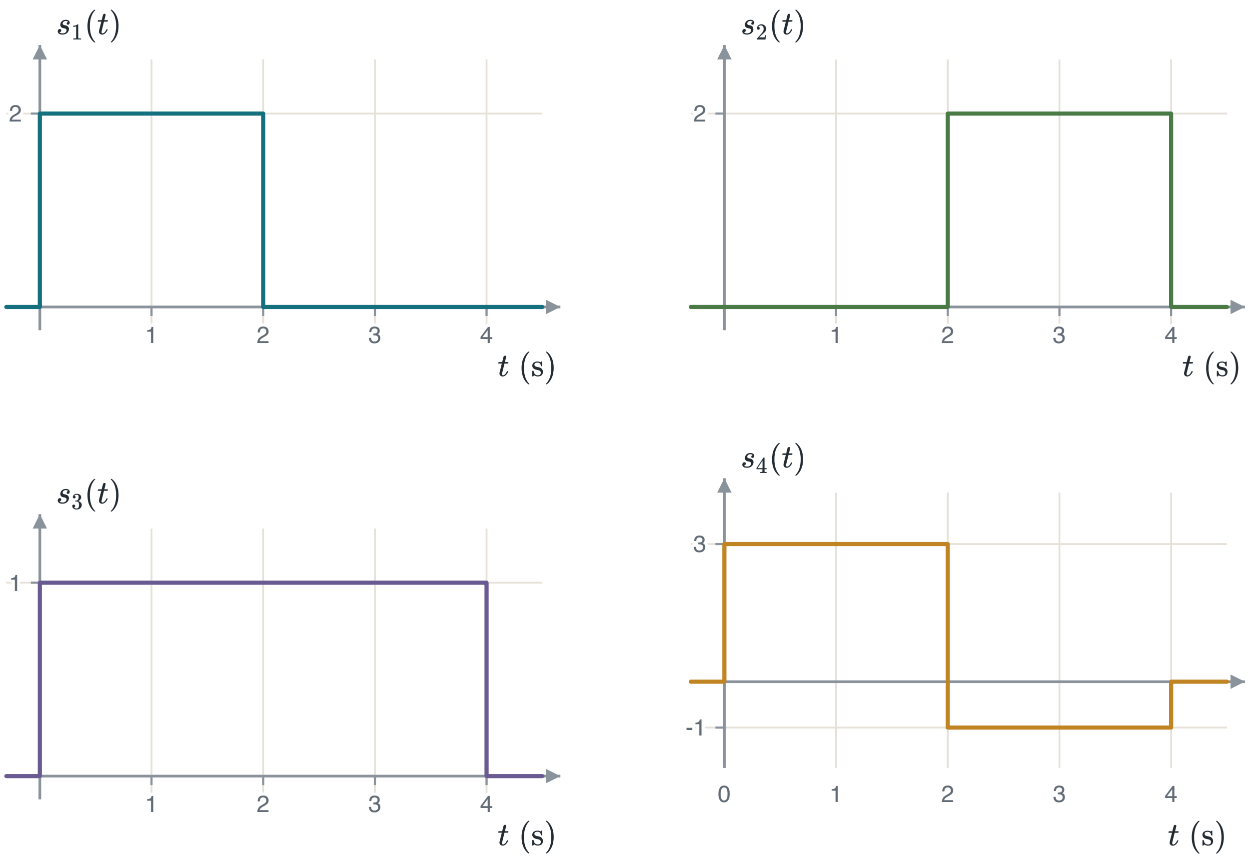
b) Calculate the symbol energies from the waveforms, and $E_{s,av}$.

c) Determine $d_{\min}$ and the number of nearest neighbours of each symbol.

d) An eight-phase set with the same $E_{s,av}$ is proposed. Which of the two sets has the larger $d_{\min}$?

D3–27 · HOW MANY DIMENSIONS A SET NEEDS

Four equiprobable messages are transmitted by the waveforms shown below, with $T = 4$ s. According to the information given above,



a) Apply the Gram–Schmidt procedure in the order s_1, s_2, s_3, s_4. How many basis functions does the set need?

b) Find the four signal vectors and the energies.

c) Determine $d_{\min}$ and the number of nearest neighbours of each symbol.

d) Draw the constellation, give $E_{s,av}$, and write $s_3(t)$ and $s_4(t)$ in terms of $s_1(t)$ and $s_2(t)$.

D3-28 · HOW MANY DIMENSIONS A SET NEEDS

Consider an M-ary modulation scheme where the equiprobable symbols have the following waveforms:

$$s_k(t) = 2(2k-5)\cos\left(2000\pi t + \frac{\pi}{3}\right), \qquad k \in \{1,2,3,4\}, \quad 0 \le t \le 1.$$

The functions $\psi_1(t) = \sqrt{2}\cos(2000\pi t)$ and $\psi_2(t) = -\sqrt{2}\sin(2000\pi t)$ are orthonormal on this interval. According to the information given above,

a) Find the signal vectors against ψ_1 and ψ_2, and draw the constellation.

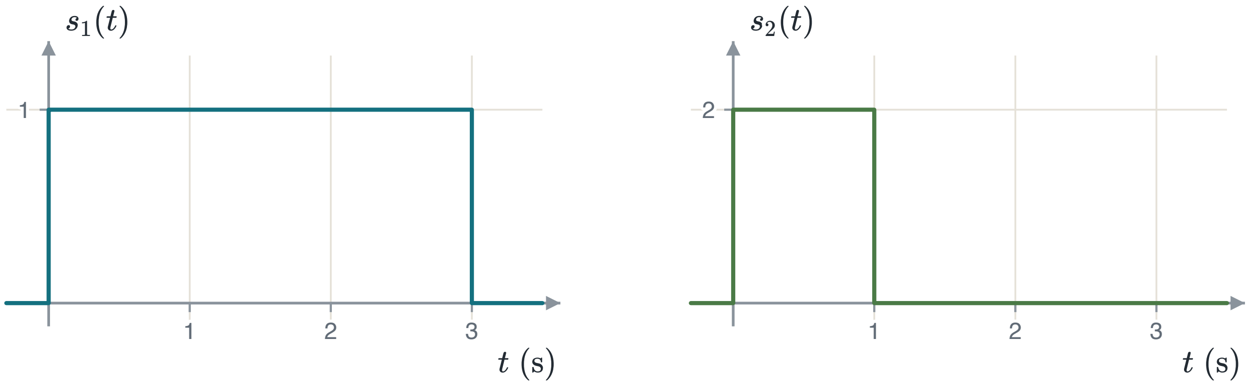
b) Show that the set needs only one dimension. Give a single unit-energy basis function $\phi(t)$ and the coordinates against it.

c) Calculate the symbol energies and $E_{s,av}$.

d) Determine $d_{\min}$ and the number of nearest neighbours, in both descriptions.

D3-29 · COMPARING TWO DESCRIPTIONS OR TWO DESIGNS

Two equiprobable messages are transmitted by the waveforms $s_1(t)$ and $s_2(t)$ shown below, with $T = 3$ s. According to the information given above,



a) Apply the Gram–Schmidt procedure starting with $s_1(t)$. Give the basis $\{\psi_1, \psi_2\}$ and the two signal vectors.

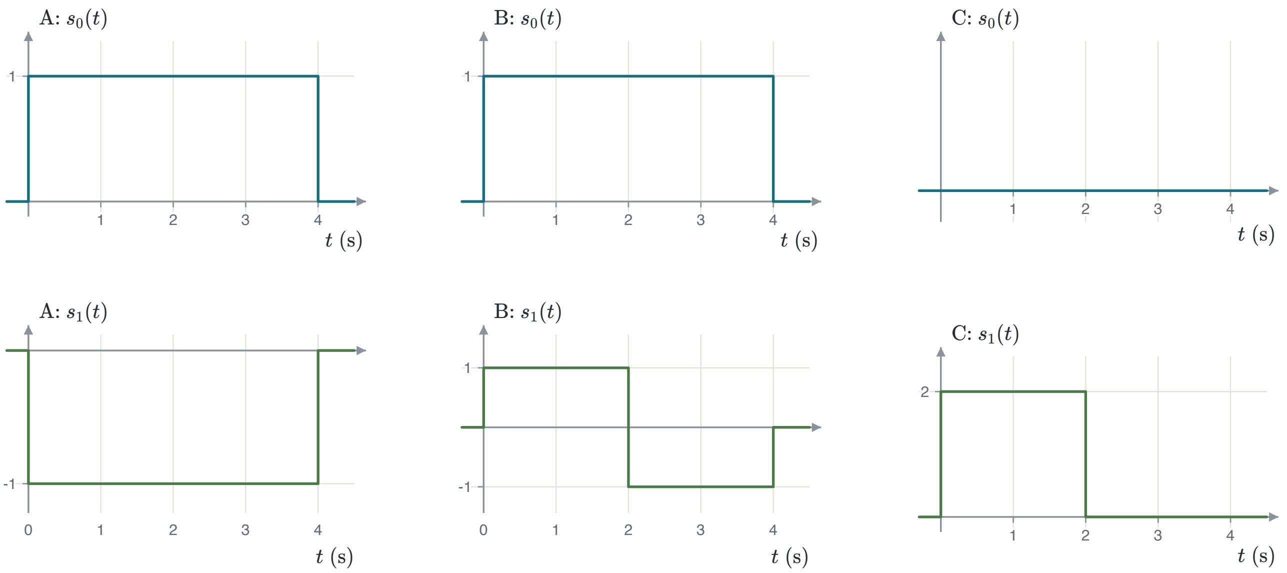
b) Apply it again starting with $s_2(t)$. Give the basis $\{\phi_1, \phi_2\}$ and the two signal vectors.

c) In each basis, calculate E_1, E_2, d_{12} and $\rho_{12} = \langle \mathbf{s}_1, \mathbf{s}_2 \rangle / \sqrt{E_1 E_2}$.

d) Draw both constellations. State what changed between them and what did not.

D3-30 · COMPARING TWO DESCRIPTIONS OR TWO DESIGNS

Three binary designs are proposed for two equiprobable messages on $[0, T]$, with $T = 4$ s. Each design uses the pair of waveforms in one column below. According to the information given above,



a) Find an orthonormal basis for each design and give its number of dimensions.

b) Find the signal vectors and $E_{s,av}$ of each design.

c) Calculate the distance d_{01} of each design.

d) Compare the designs by $d_{01}^2/E_{s,av}$. By how many decibels does the best design exceed the others?

MODULE

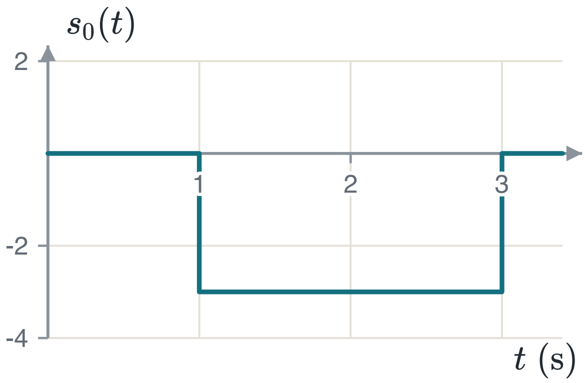
4

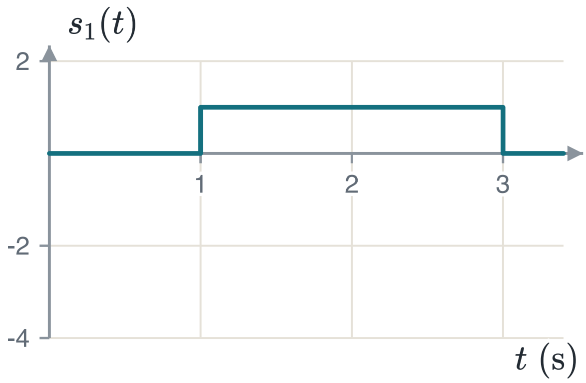
The Optimal Receiver in AWGN

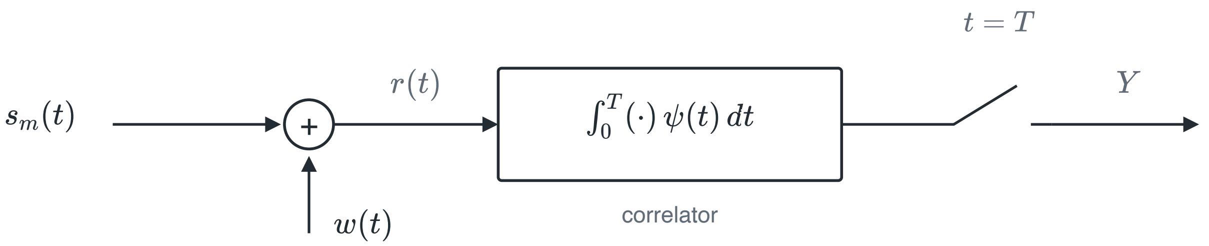
30 questions on the optimal receiver in awgn. Write your reasoning in the space under each one.

D4-01 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

Two equiprobable messages, "0" and "1", are sent over an additive white Gaussian noise channel. The two-sided noise power spectral density is $N_0/2 = 2$ W/Hz. Message m is carried by the waveform $s_m(t)$ drawn below, with $T = 3$ s. These signals are passed through the following correlator-type demodulator. Here $\psi(t)$ is the unit-energy basis function of the two signals, and $w(t)$ is the white Gaussian noise. According to the information given above,







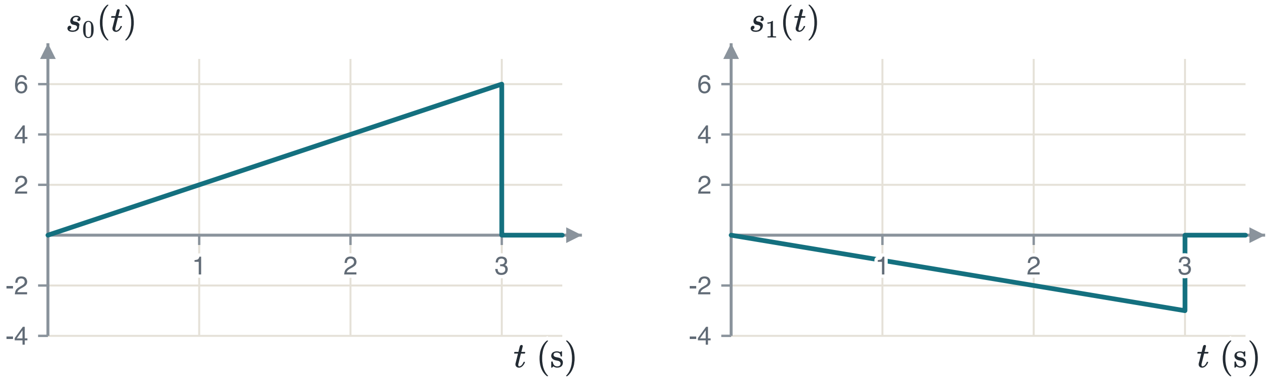
a) Find $\psi(t)$ and the coordinates s_0 and s_1 of the two signals on it.

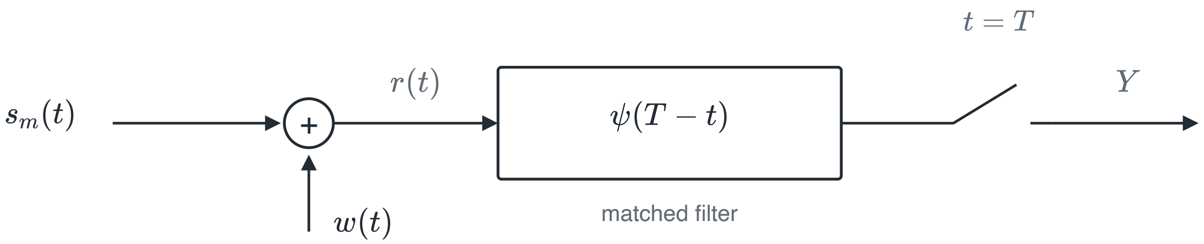
b) Determine the conditional probability density functions $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold λ and calculate the average probability of bit error P_b.

D4-02 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

In an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 9$ W/Hz, two equiprobable messages, "0" and "1", are transmitted. Their waveforms $s_0(t)$ and $s_1(t)$ are drawn below. These signals are passed through the following matched-filter type demodulator, sampled at $t = T = 3$ s. Here $\psi(t)$ is the unit-energy basis signal that represents both waveforms and is also the impulse response of the matched filter. According to the information given above,





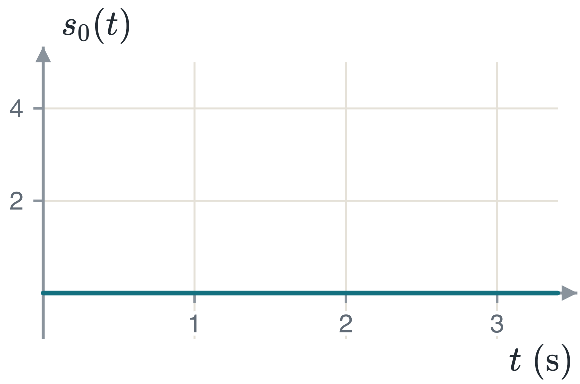
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

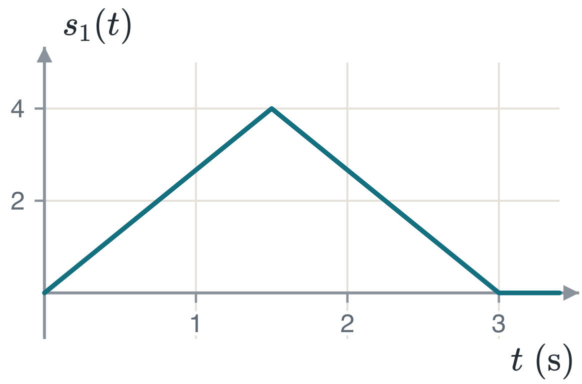
b) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

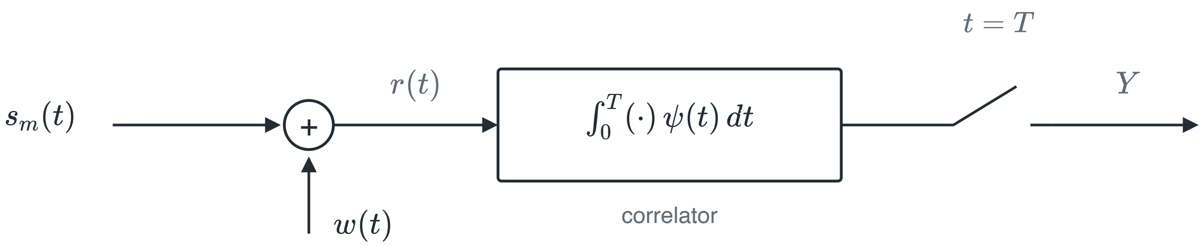
c) Find the optimal decision threshold λ and calculate the average probability of bit error P_b. Use a Q-function table.

D4-03 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

Two equiprobable messages are sent over an additive white Gaussian noise channel with two-sided noise power spectral density $N_0/2 = 0.64$ W/Hz. Message "0" is sent by $s_0(t) = 0$. Message "1" is sent by the triangular pulse $s_1(t)$ drawn below, with $T = 3$ s. The received signal passes through the following correlator-type demodulator. According to the information given above,







a) Find the basis function $\psi(t)$, the coordinates s_0 and s_1, and the energies E_0 and E_1.

b) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold λ and the average probability of bit error P_b.

d) The receiver is rebuilt to compare $\int_0^T r(t)s_i(t)\,dt - E_i/2$ for $i = 0, 1$ and pick the larger. Show that it makes the same decisions as part (c).

D4-04 · UNION BOUND AND EQUAL-ENERGY COMPARISON

A symbol of length $T = 2$ s has four slots of 0.5 s. The pulse in slot i is $p_i(t) = 2$ for $0.5(i-1) \le t < 0.5i$, and zero elsewhere, with $i = 1, \ldots, 4$. Two sets of four equally likely signals are built from these pulses. Set A puts one pulse in one slot, $s_i(t) = p_i(t)$. Set B puts pulses in two slots:

$$s_1 = p_1 + p_2, \quad s_2 = p_3 + p_4, \quad s_3 = p_1 + p_3, \quad s_4 = p_2 + p_4.$$

The channel adds white Gaussian noise with power spectral density $N_0/2$. Let E_p be the energy of one pulse. The receiver uses the basis $\psi_i(t) = p_i(t)/\sqrt{E_p}$ and forms $r_i = \int_0^T r(t)\psi_i(t)\,dt$. According to the information given above,

- **a)** Find E_p, the four signal vectors of each set, and the energy per bit E_b of each set.
- **b)** Find every pairwise distance in each set in terms of its E_b. Write the union bound on the symbol error probability P_e of each set as a function of E_b/N_0.
- **c)** Find the penalty of set B in decibels at high E_b/N_0. Evaluate both union bounds at $E_b/N_0 = 4$.
- **d)** Show that the ML receiver for set B needs only the signs of $Z_1 = r_1 - r_4$ and $Z_2 = r_2 - r_3$. Find the exact P_e of set B and evaluate it at $E_b/N_0 = 4$.

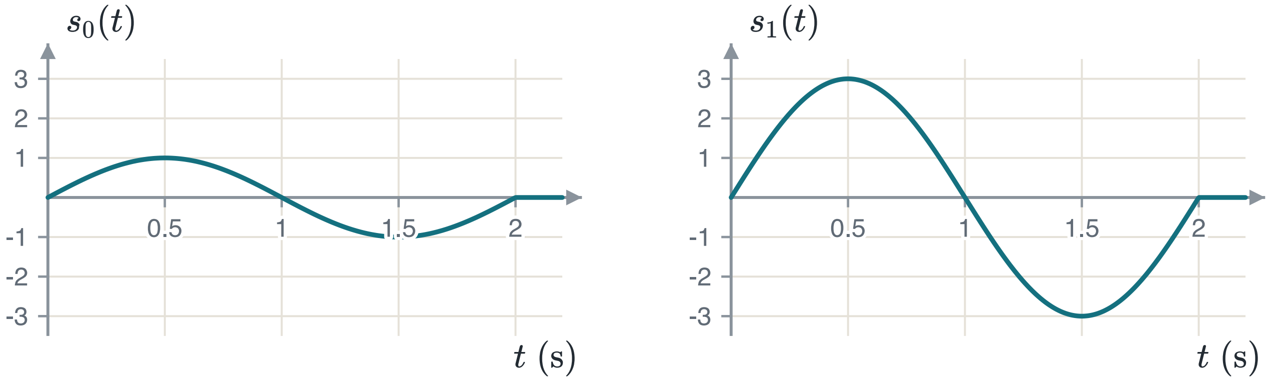
D4-05 · A RECEIVER WITH MISPLACED THRESHOLDS

A 4-PAM receiver expects the noiseless samples -6, -2, 2 and 6. It decides with the fixed thresholds -4 , 0 and 4. The four symbols are equally likely. The noise on the sample is Gaussian with mean 0 and variance 1. A faulty gain stage scales the noiseless samples by a factor g, so they arrive at $\pm 2g$ and $\pm 6g$. The thresholds and the noise variance do not change. The gain stage divides its input by $\sqrt{\hat{P}}$, where $\hat{P}$ is its estimate of the received power P. According to the information given above,

- **a)** For $g = 0.85$, sketch the thresholds and the four noiseless samples. Give the distance from each sample to each threshold beside it.
- **b)** For $g = 0.85$, find the conditional error probability of an inner and of an outer symbol, and the average symbol error probability P_e.
- **c)** Repeat part (b) for $g = 1.15$.
- **d)** For each g, give the loss in decibels at high signal-to-noise ratio against thresholds at the midpoints of the scaled samples. Say which gain is worse, and whether $\hat{P}$ is too large or too small.

D4-06 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

Two equiprobable messages, "0" and "1", are sent over an additive white Gaussian noise channel with $N_0/2 = 0.36$ W/Hz. The waveforms are $s_0(t) = \sin(\pi t)$ and $s_1(t) = 3\sin(\pi t)$ for $0 \le t \le 2$ s, drawn below. The received signal passes through a correlator-type demodulator with $T = 2$ s. According to the information given above,



a) Find $\psi(t)$, the coordinates s_0 and s_1, and the energies E_0 and E_1.

b) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold λ and calculate P_b.

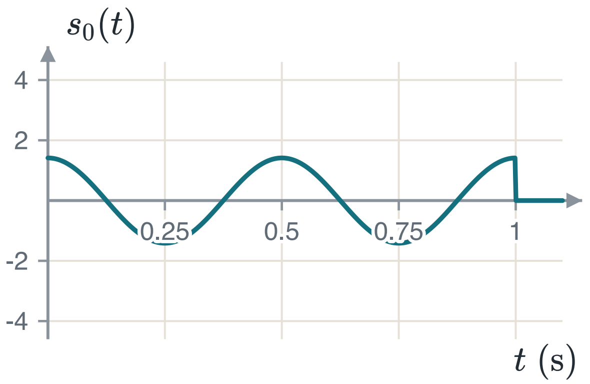
d) A designer builds a receiver that picks the larger of $\int_0^T r(t)s_i(t)\,dt$ and leaves out $-E_i/2$. Find its P_b
.

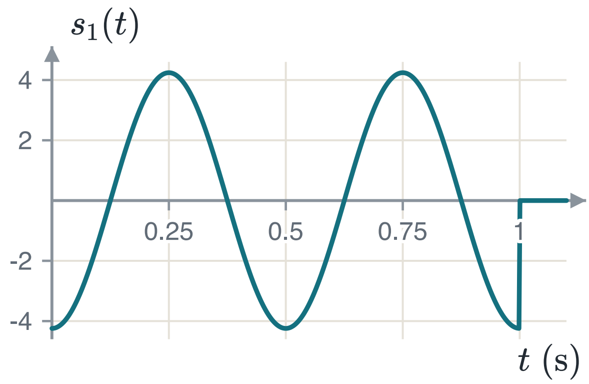
D4-07 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

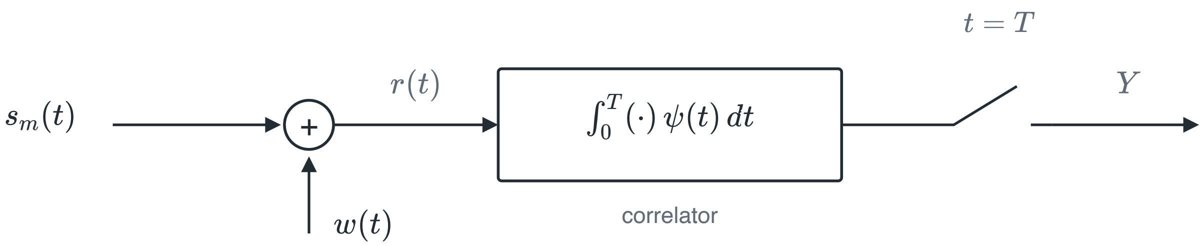
Two equiprobable messages are sent over an additive white Gaussian noise channel with $N_0/2 = 0.49$ W/Hz. They are transmitted by

$$s_0(t) = \sqrt{2}\cos(4\pi t), \qquad s_1(t) = -3\sqrt{2}\cos(4\pi t), \qquad 0 \le t \le 1 \text{ s}.$$

These signals are passed through a correlator-type demodulator with $T = 1$ s, drawn below. According to the information given above,







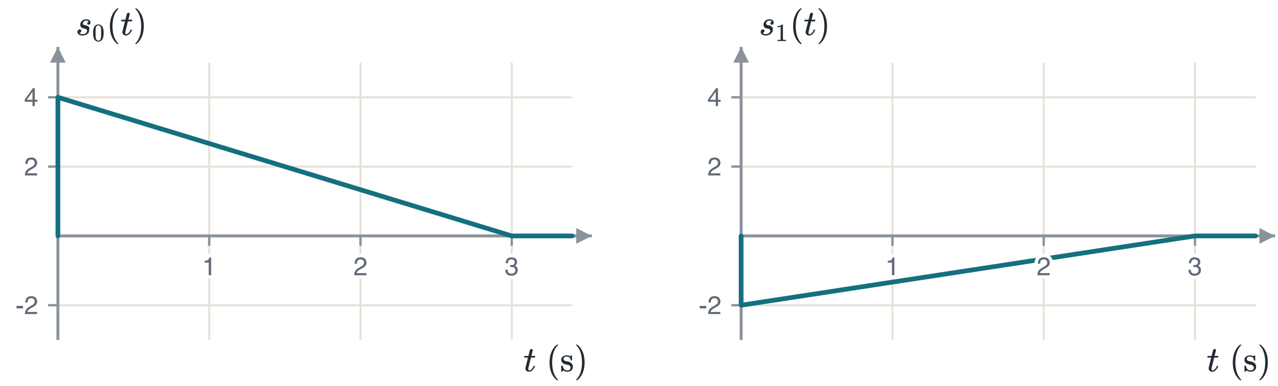
a) Find the unit-energy basis $\psi(t)$ and the coordinates s_0 and s_1.

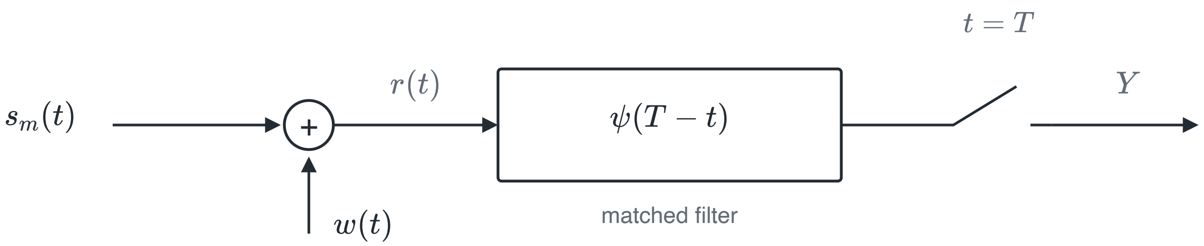
b) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold λ and calculate the average probability of bit error P_b.

D4-08 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

In an additive white Gaussian noise channel with $N_0/2 = 4$ W/Hz, "0" and "1" bits occur with probabilities 0.6 and 0.4. They are transmitted by the waveforms $s_0(t)$ and $s_1(t)$ drawn below. These signals are passed through the following matched-filter type demodulator, sampled at $t = T = 3$ s. According to the information given above,





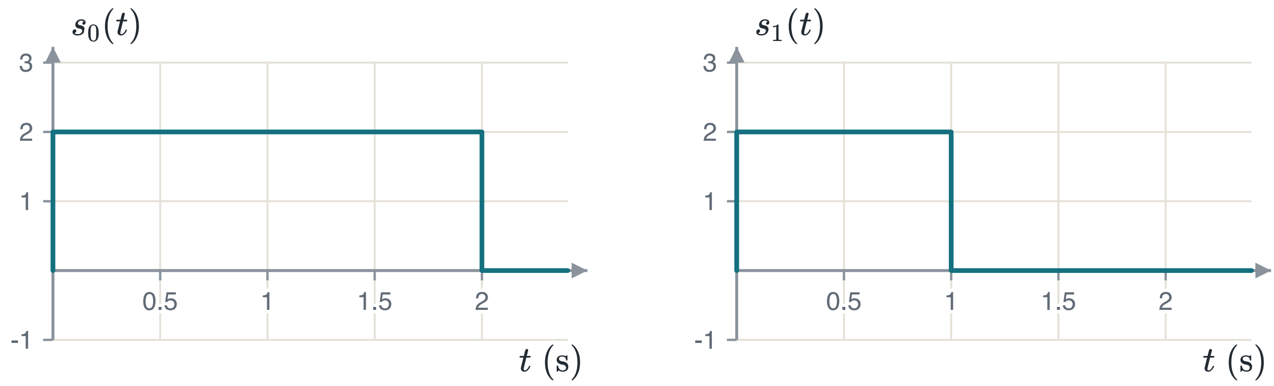
a) Plot the impulse response of the matched filter, $h(t) = \psi(T - t)$.

b) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold λ and calculate the average probability of bit error P_b.

D4-09 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

Two equiprobable messages are sent over an additive white Gaussian noise channel with $N_0/2 = 0.2$ W/Hz. The waveforms are $s_0(t) = 2$ for $0 \le t < 2$, and $s_1(t) = 2$ for $0 \le t < 1$ and $s_1(t) = 0$ for $1 \le t < 2$. They are drawn below, with $T = 2$ s. The receiver uses two correlators, with the basis functions of part (a). According to the information given above,



a) Apply Gram–Schmidt, starting from $s_1(t)$, to find $\psi_1(t)$, $\psi_2(t)$ and the vectors $\mathbf{s}_0$ and $\mathbf{s}_1$.

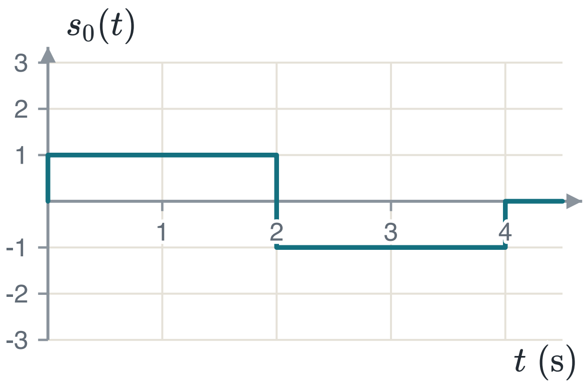
b) Find the ML decision rule and draw the decision regions.

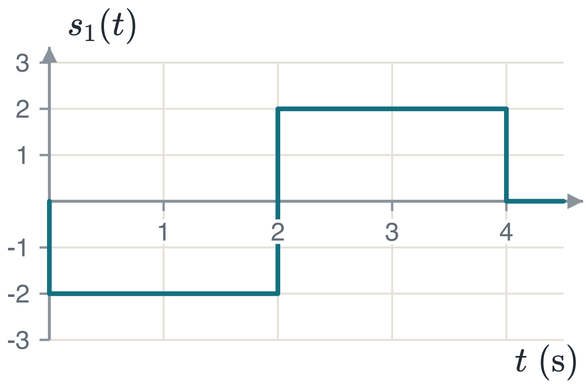
c) Calculate the average probability of bit error P_b.

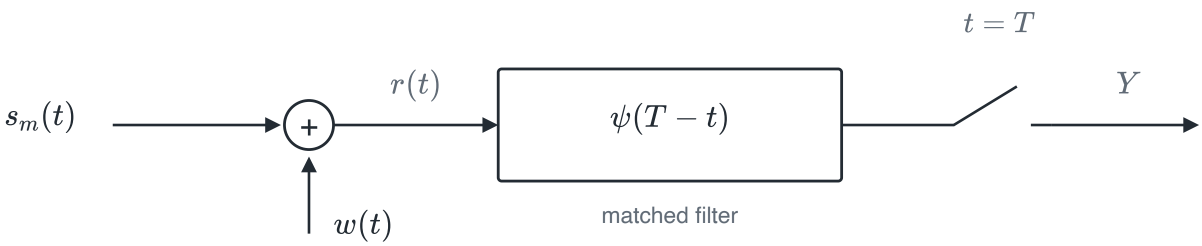
d) Show that one correlator with $s_0(t) - s_1(t)$ makes the same decisions, and give its threshold.

D4-10 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

In an additive white Gaussian noise channel with noise power spectral density $N_0/2 = 9$ W/Hz, two equiprobable messages are sent by the waveforms drawn below. These signals are passed through the following matched-filter type demodulator, sampled at $t = T = 4$ s. According to the information given above,







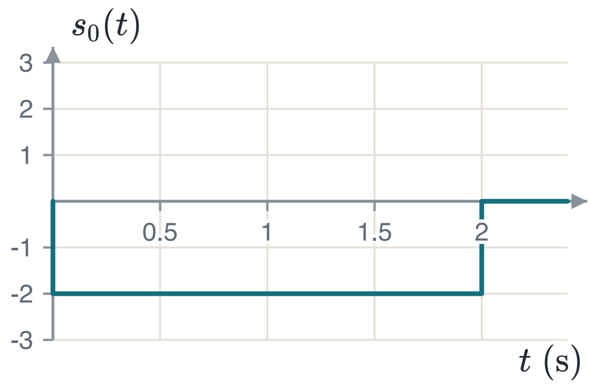
a) Plot the impulse response of the matched filter, $h(t) = \psi(T-t)$.

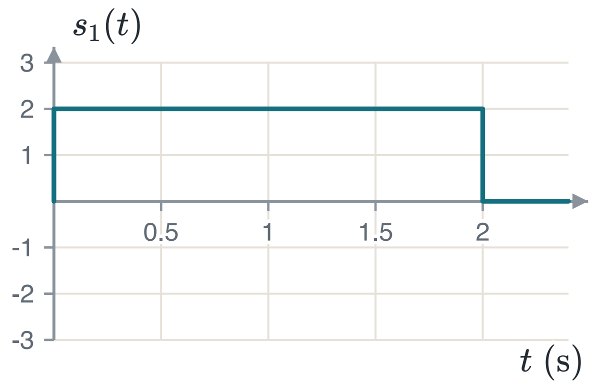
b) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

c) Find the optimal decision threshold λ and calculate the average probability of bit error P_b.

D4-11 · TWO WAVEFORMS, THE OPTIMAL RECEIVER AND THE BIT ERROR PROBABILITY

A binary link uses $s_1(t) = 2$ and $s_0(t) = -2$ for $0 \le t < 2$ s, with equal priors. The noise is white and Gaussian with an unknown two-sided power spectral density $N_0/2$. The receiver is the optimal correlator receiver with $T = 2$ s. The link must reach $P_b \le 1.0 \times 10^{-3}$. A Q table gives $Q(3.09) = 1.001 \times 10^{-3}$ and $Q(3.10) = 0.968 \times 10^{-3}$. According to the information given above,





a) Find the coordinates s_0 and s_1, and write P_b as a function of $N_0/2$.

b) Find the largest $N_0/2$ that meets the requirement.

c) The link is changed to on-off signalling, $s_0(t) = 0$ and $s_1(t) = A$ on $[0, 2)$, with the same average energy per bit. Find A, the largest $N_0/2$ now, and the loss in decibels.

D4-12 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

Assume that in a binary pulse amplitude modulation (PAM) communication system, "0" and "1" bits occur with probabilities 0.4 and 0.6. At the output of the matched filter the sample is

$$Y = \begin{cases} 2 + N, & \text{if "1" is sent} \\ -2 + N, & \text{if "0" is sent} \end{cases}.$$

Here N is a Gaussian random variable with mean 0 and variance 2.25. According to this information,

a) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

b) Find the optimal decision threshold λ.

c) Calculate the average probability of bit error P_b as a number. Use a Q-function table.

D4-13 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

Assume that in a binary PAM communication system, "0" and "1" bits occur with probabilities 0.6 and 0.4. Consider a signal detector with the input

$$Y = \begin{cases} 5 + N, & \text{if "1" is sent} \\ N, & \text{if "0" is sent} \end{cases}.$$

Here N is a Gaussian random variable with mean 0 and variance 4. According to this information,

a) Find the optimal decision threshold λ.

b) Calculate the average probability of bit error P_b as a number. Use a Q-function table.

c) A simpler detector uses the midpoint 2.5 as its threshold. Calculate its P_b and compare.

D4-14 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

In a binary PAM system the matched-filter output is $Y = 1 + N$ when "1" is sent and $Y = -2 + N$ when "0" is sent. N is Gaussian with mean 0 and variance 1. The bits "0" and "1" occur with probabilities 0.75 and 0.25. According to this information,

a) Determine the conditional PDFs $f_Y(y \mid 1)$ and $f_Y(y \mid 0)$.

b) Find the ML threshold and the P_b of a detector that uses it.

c) Find the optimal (MAP) decision threshold λ.

d) Calculate P_b of the MAP detector as a number, and compare.

D4-15 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

Bits are sent by antipodal signalling with energy $E_b = 2.25$ J per bit. The two signal points are $s_0 = -\sqrt{E_b}$ and $s_1 = +\sqrt{E_b}$ on one basis function. The noise is white and Gaussian with $N_0 = 0.5$ W/Hz. The bit "0" occurs with probability 0.8 and "1" with probability 0.2. According to this information,

a) Find the distance d between the two points and the P_b of the ML detector.

b) Find the MAP boundary: its distance μ from s_0, and the threshold λ.

c) Calculate the conditional error probabilities $P(e \mid 0)$ and $P(e \mid 1)$ of the MAP detector.

d) Calculate P_b of the MAP detector and compare it with part (a).

D4-16 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

A three-level PAM system sends $Y = s + N$ with $s \in \{-2,\ 0,\ 2\}$. The priors are $P(-2) = P(2) = 0.25$ and $P(0) = 0.5$. N is Gaussian with mean 0 and variance 0.5. According to this information,

a) Write the three conditional PDFs $f_Y(y \mid s)$.

b) Find the two optimal (MAP) thresholds $\lambda_1 < \lambda_2$.

c) Calculate the average symbol error probability P_e as a number.

d) Calculate P_e of the ML detector, with thresholds ± 1, and compare.

D4-17 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

In a binary PAM system the bit "1" is twice as likely as the bit "0". The matched-filter output is $Y = 1 + N$ when "1" is sent and $Y = -1 + N$ when "0" is sent. N is Gaussian with mean 0 and variance 0.25. According to this information,

a) Find the two priors and the optimal decision threshold λ.

b) Calculate the average probability of bit error P_b as a number.

c) The noise variance rises to 1. Find the new threshold and P_b, and say how far λ moved.

D4-18 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

In a binary PAM system the matched-filter output is $Y = 2 + N$ for "1" and $Y = -2 + N$ for "0". N is Gaussian with mean 0 and variance 4. The bit "0" occurs with probability 0.9 and "1" with probability 0.1. According to this information,

a) For the sample $y = 0.5$, compute $P_0 f_Y(y \mid 0)$ and $P_1 f_Y(y \mid 1)$. Give the MAP and the ML decisions.

b) Find the MAP threshold λ.

c) Calculate P_b of the MAP detector as a number.

d) Show that $P(e \mid 1)$ exceeds one half, and compare P_b with the ML detector.

D4-19 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

In a binary PAM system the matched-filter output is $Y = 1 + N$ for "1" and $Y = -1 + N$ for "0". N is Gaussian with mean 0 and variance 0.5. The priors are not known. The MAP detector of this system uses the threshold $\lambda = -0.25$. According to this information,

a) Find the priors P_0 and P_1.

b) Calculate P_b of this MAP detector as a number.

c) Calculate P_b of the ML detector with the same priors, and compare.

D4-20 · BINARY PAM WITH UNEQUAL PRIORS: THE MAP THRESHOLD

Two messages use $s_0(t) = 2\sqrt{2}\cos(2\pi t)$ and $s_1(t) = 2\sqrt{2}\sin(2\pi t)$ for $0 \le t \le 1$ s. The channel adds white Gaussian noise with $N_0/2 = 0.5$ W/Hz. The priors are $P(s_0) = 0.75$ and $P(s_1) = 0.25$. The receiver has two correlators with $\psi_1(t) = \sqrt{2}\cos(2\pi t)$ and $\psi_2(t) = \sqrt{2}\sin(2\pi t)$. According to the information given above,

a) Find the signal vectors $\mathbf{s}_0$ and $\mathbf{s}_1$, and write the MAP metric of each.

b) Find the MAP decision boundary and draw the two decision regions.

c) The observation is $\mathbf{r} = (0.9,\ 1.1)$. Give the MAP and the ML decisions.

d) Calculate P_b of the MAP detector and compare it with the ML detector.

D4-21 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the waveforms

$$s_k(t) = \sqrt{6}\cos\left(2000\pi t + \frac{(2k-1)\pi}{6}\right), \qquad k \in \{1, \ldots, 6\},\ 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

a) Choose an orthonormal basis, find the coordinates of the six points, and give $E_{s,\mathrm{avg}}$.

b) Draw the signal constellation and the optimal decision regions.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\mathrm{avg}}/N_0$.

D4-22 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$\begin{aligned} s_1(t) &= 0 \\ s_2(t) &= 2\sqrt{2}\cos(2000\pi t) \\ s_3(t) &= 4\cos\left(2000\pi t + \frac{\pi}{4}\right) \\ s_4(t) &= 2\sqrt{2}\cos\left(2000\pi t + \frac{\pi}{2}\right) \end{aligned}$$

all for $0 \le t \le 1$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

a) Find the four signal points and $E_{s,\text{avg}}$.

b) Draw the signal constellation and the optimal decision regions.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$.

d) Moving the constellation so that its centre is at the origin keeps P_e. How much energy does it save, in decibels?

D4-23 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the waveforms

$$s_k(t) = (2k-5)\sqrt{2}\cos(4000\pi t), \qquad k \in \{1,2,3,4\},\ 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard additive white Gaussian noise (AWGN) channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

a) Find the basis, the four signal points and $E_{s,\text{avg}}$.

b) Draw the signal constellation and the optimal decision regions.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$.

D4-24 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the waveforms

$$s_k(t) = 2\cos\left(1000\pi t + \frac{(2k-1)\pi}{8}\right), \qquad k \in \{1, \ldots, 8\},\ 0 \le t \le 2.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

a) Choose an orthonormal basis and find the signal points and $E_{s,\mathrm{avg}}$.

b) Draw the signal constellation and the optimal decision regions.

c) Determine the nearest-neighbour approximation of P_e as a function of $E_{s,\mathrm{avg}}/N_0$.

d) Evaluate it at $E_{s,\mathrm{avg}}/N_0 = 20$.

D4-25 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the waveforms

$$s_{a,b}(t) = a\sqrt{2}\cos(2000\pi t) - b\sqrt{2}\sin(2000\pi t), \qquad 0 \le t \le 1,$$

with $a \in \{-3, -1, 1, 3\}$ and $b \in \{-1, 1\}$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

a) Find the signal points and $E_{s,\mathrm{avg}}$.

b) Draw the signal constellation and the optimal decision regions.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\mathrm{avg}}/N_0$.

D4-26 · A DECISION WITH AN ERASURE ZONE

A QPSK receiver works on the two correlator outputs $\mathbf{y} = (y_1, y_2)$. The four equally likely signal points are $\mathbf{s}_1 = (2, 2)$, $\mathbf{s}_2 = (-2, 2)$, $\mathbf{s}_3 = (-2, -2)$ and $\mathbf{s}_4 = (2, -2)$. Each coordinate carries independent Gaussian noise with mean 0 and variance $N_0/2 = 0.64$. The receiver does not decide when the observation lies close to a boundary. If $|y_1| < 0.4$ or $|y_2| < 0.4$, it puts out an erasure, a mark that means "no decision". Otherwise it decides the point in the quadrant of $\mathbf{y}$. Let d be the distance between neighbouring points, d_1 the width of each erasure strip, and $\alpha = d_1/d$. The energy per bit is $E_b = E_{s,\text{avg}}/2$. According to the information given above,

- **a)** Find d, d_1, α, E_b and E_b/N_0. Draw the decision regions and the erasure zone.
- **b)** Use the intelligent union bound to approximate the symbol error probability p and the erasure probability q. Write each as a function of E_b/N_0 and α, then evaluate it.
- **c)** Find p and q exactly, as products of probabilities of the two coordinates, and evaluate them.
- **d)** Find the symbol error probability of the ordinary QPSK receiver, which has no erasure zone. State what the zone gains and what it costs.

D4-27 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the waveforms

$$s_k(t) = 2\cos\left(2000\pi t + \frac{2\pi k}{3} + \frac{\pi}{2}\right), \qquad k \in \{0, 1, 2\},\ 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

- **a)** Find the three signal points and $E_{s,\text{avg}}$.
- **b)** Draw the signal constellation and the optimal decision regions.
- **c)** Determine the nearest-neighbour approximation of P_e as a function of $E_{s,\text{avg}}/N_0$.
- **d)** Evaluate it at $E_{s,\text{avg}}/N_0 = 6$.

D4-28 · M-ARY CONSTELLATIONS: DECISION REGIONS AND THE NEAREST-NEIGHBOUR APPROXIMATION

Consider an M-ary modulation scheme where the equally probable symbols have the waveforms $s_1(t) = 0$ and

$$s_k(t) = 2\sqrt{2}\cos\left(2000\pi t + \frac{(k-2)\pi}{3}\right), \qquad k \in \{2, \ldots, 7\},\ 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. According to the information given above,

a) Find the seven signal points and $E_{s,\mathrm{avg}}$.

b) Draw the signal constellation and the optimal decision regions.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\mathrm{avg}}/N_0$.

D4-29 · UNION BOUND AND EQUAL-ENERGY COMPARISON

Two designs carry three bits per symbol with the same average symbol energy $E_{s,\mathrm{avg}}$ over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. Design A is 8-PSK: eight equally likely points on a circle, 45° apart. Design B is the rectangle of points $c(a, b)$ with $a \in \{\pm 1, \pm 3\}$, $b \in \{\pm 1\}$ and a scale $c > 0$. According to the information given above,

a) Find $d^2_{\min}$ of each design in terms of $E_{s,\mathrm{avg}}$.

b) Find $N_{\min}$ of each design.

c) Evaluate the nearest-neighbour approximation of P_e for both at $E_{s,\mathrm{avg}}/N_0 = 20$.

d) Say which design is better, and give its gain in $d^2_{\min}$ in decibels.

D4-30 · UNION BOUND AND EQUAL-ENERGY COMPARISON

Three equally likely messages use $s_1(t) = 0$, $s_2(t) = 2$ for $0 \le t < 1$ (zero after), and $s_3(t) = 3$ for $1 \le t < 2$ (zero before). $T = 2$ s. The channel adds white Gaussian noise with $N_0/2 = 0.25$ W/Hz. The receiver is the optimal ML receiver. According to the information given above,

a) Find the basis, the three signal points, and draw the decision regions.

b) Find all pairwise distances, $d_{\min}$ and $N_{\min}$.

c) Evaluate the union bound on the symbol error probability P_e.

d) Evaluate the nearest-neighbour approximation and the minimum-distance bound, and put the three numbers in order.

MODULE

5

Digital Modulation Methods

30 questions on digital modulation methods. Write your reasoning in the space under each one.

D5-01 · EQUAL-ENERGY PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = \sqrt{18}\cos\left(3000\pi t + \frac{\pi(2k+1)}{6}\right), \quad k \in \{1, \dots, 6\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Evaluate the approximation of part (b) at $E_{s,\text{avg}}/N_0 = 18$.

D5-02 · EQUAL-ENERGY PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = \sqrt{24}\cos\left(2000\pi t + \frac{\pi(4k-3)}{6}\right), \quad k \in \{1, 2, 3\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$.

c) How many bits does one symbol carry? Rewrite the result of part (b) as a function of E_b/N_0.

D5-03 · EQUAL-ENERGY PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = 2\cos\left(4000\pi t + \frac{(2k-1)\pi}{8}\right), \quad k \in \{1, \ldots, 8\}, \quad 0 \le t \le 2.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Find $E_{s,\text{avg}}$, the number of bits per symbol and the average energy per bit E_b.

b) Draw the signal constellation and the optimal decision regions for this signal constellation.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, and then as a function of E_b/N_0.

D5-04 · AMPLITUDE SETS ON ONE CARRIER

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = 2(2k-5)\cos(5000\pi t), \quad k \in \{1, 2, 3, 4\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Evaluate the approximation of part (b) at $E_{s,\text{avg}}/N_0 = 22.5$.

D5-05 · AMPLITUDE SETS ON ONE CARRIER

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = 3\sqrt{2}\,(k-1)\cos(3000\pi t), \quad k \in \{1,2,3\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) The set is shifted so that its points are symmetric about the origin, with the same $d_{\min}$. Find the saving in $E_{s,\text{avg}}$ in decibels.

D5-06 · AMPLITUDE SETS ON ONE CARRIER

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = 2(k-3)\cos(4000\pi t), \quad k \in \{1,\dots,5\}, \quad 0 \le t \le 0.5.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Find $E_{s,\text{avg}}$ and the number of bits carried by one symbol.

b) Draw the signal constellation and the optimal decision regions for this signal constellation.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$. Evaluate it at $E_{s,\text{avg}}/N_0 = 36$.

D5-07 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_1(t) = 0,$$
$$s_k(t) = 2\sqrt{2}\cos\left(2000\pi t + \frac{\pi}{2} + \frac{2\pi(k-2)}{3}\right), \quad k \in \{2,3,4\},$$

all on $0 \le t \le 1$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Evaluate the approximation of part (b) at $E_{s,\text{avg}}/N_0 = 6$.

D5-08 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_{mn}(t) = \sqrt{2}\big[(2m-5)\cos(3000\pi t) - (2n-3)\sin(3000\pi t)\big],$$

with $m \in \{1,2,3,4\}$, $n \in \{1,2\}$ and $0 \le t \le 1$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Evaluate the approximation of part (b) at $E_{s,\text{avg}}/N_0 = 27$.

D5-09 · RATE, BAND AND POWER OF SEVERAL SCHEMES

A wireless link sends 16-QAM at $R_b = 48$ Mb/s on a carrier at $f_c = 2.4$ GHz. The pulses are raised cosines with roll-off $\alpha = 0.25$. On a carrier the occupied band is $B = (1+\alpha)R_s$, where R_s is the symbol rate. The transmit power is 250 mW. The noise level N_0 and the path to the receiver are the same for every scheme below.

a) Find the symbol rate R_s, the occupied band B and the frequency interval that the signal occupies.

b) QPSK is sent in the same band with the same roll-off. Find its symbol rate and its bit rate.

c) The QPSK link must give the same symbol error probability at high signal-to-noise ratio. Use the nearest-neighbour approximation and ignore the difference in $N_{\min}$. Find the QPSK transmit power and the saving in decibels.

d) 64-QAM is to carry the same 48 Mb/s with the same roll-off. Find its band and the transmit power it needs for the same error. Give R_b/B for all three schemes.

D5-10 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = A_k \cos\Big(2000\pi t + \frac{(k-1)\pi}{4}\Big), \quad k \in \{1,\ldots,8\}, \quad 0 \le t \le 1,$$

where $A_k = \sqrt{2}$ for odd k and $A_k = 4$ for even k. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

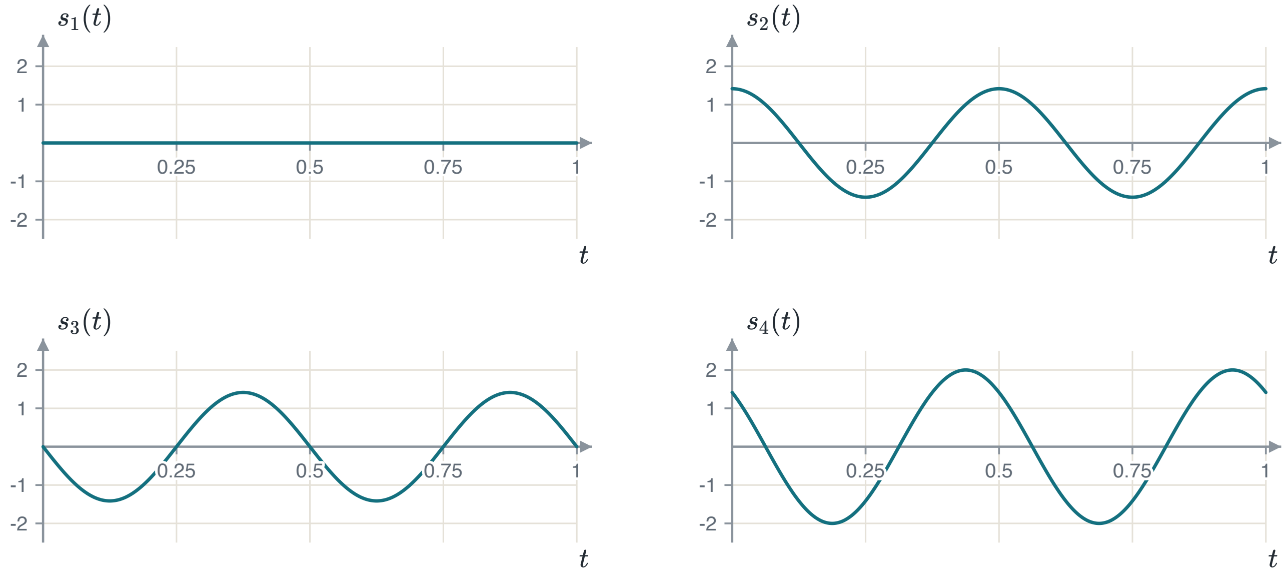
a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Which symbols have no neighbour at $d_{\min}$? Evaluate the approximation at $E_{s,\text{avg}}/N_0 = 40.5$.

D5-11 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the four equally probable symbols have the waveforms drawn below, each on $0 \leq t \leq 1$. Each waveform is zero or a sinusoid at 2 Hz. The peak value is $\sqrt{2}$ for s_2 and s_3 and 2 for s_4. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.



a) Write each waveform in the form $A\cos(4\pi t + \theta)$ and find its signal-space coordinates.

b) Draw the signal constellation and the optimal decision regions for this signal constellation.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$.

d) The four points are moved together so that their centre is at the origin. Find the saving in $E_{s,\text{avg}}$ in decibels.

D5-12 · EQUAL-ENERGY PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = 2\sqrt{3}\cos\left(2000\pi t + \frac{\pi}{6} + \frac{(k-1)\pi}{2}\right), \quad k \in \{1,2,3,4\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) With Gray labelling, one symbol error costs about one bit error. Find E_b, then compare the bit error probability with binary PSK at $E_b/N_0 = 4.5$.

D5-13 · AMPLITUDE SETS ON ONE CARRIER

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = \sqrt{2}\,(2k-9)\cos(2000\pi t), \quad k \in \{1,\dots,8\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Find $E_{s,\text{avg}}$, the number of bits per symbol and the average energy per bit E_b.

b) Draw the signal constellation and the optimal decision regions for this signal constellation.

c) Determine the nearest-neighbour approximation as a function of $E_{s,\text{avg}}/N_0$ and of E_b/N_0. Find the E_b/N_0 in decibels that makes the Q argument 3.00, and the resulting P_e.

D5-14 · TONE SPACING FOR FSK

Two tones are sent on $0 \le t \le T$ with $T = 2$ ms:

$$s_0(t) = A\cos(2\pi f_0 t + \varphi_0), \qquad s_1(t) = A\cos(2\pi f_1 t + \varphi_1),$$

with $f_0 = 20$ kHz and $f_1 = f_0 + \Delta f$, $\Delta f > 0$. Both tones have energy $E = A^2T/2$. Neglect every term at the sum frequency $f_0 + f_1$. The correlation coefficient is $\rho = \langle s_0, s_1 \rangle / E$.

a) Find $\langle s_0, s_1 \rangle$ as a function of Δf and $\Delta\varphi = \varphi_1 - \varphi_0$. Evaluate ρ for $\Delta f = 125$ Hz and equal phases.

b) With equal phases, find the smallest Δf that makes the two tones orthogonal.

c) Find the smallest Δf that makes the tones orthogonal for every $\Delta\varphi$. Show that the spacing of part (b) fails for $\Delta\varphi = 90°$.

d) An 8-FSK set uses tones of this kind with the same T. Find its bit rate, and its band with the spacing of part (b) and with that of part (c).

D5-15 · LINK BUDGET WHAT-IFS

A line-of-sight radio link is designed for a range $d_0 = 5$ km at a carrier frequency $f_c = 1.5$ GHz. It sends QPSK with Gray labels at $R_b = 8$ Mb/s. The pulses are raised cosines with roll-off $\alpha = 0.25$, so the occupied band is $B = (1+\alpha)R_s$, where R_s is the symbol rate. The target $P_b = 10^{-5}$ needs $E_b/N_0 = 9.6$ dB. The transmit power is $P_t = 20$ dBm, each antenna has a gain of 10 dBi, and the receiver noise figure is $\mathrm{NF} = 4$ dB. Take $kT_0 = -174$ dBm/Hz, $c = 3 \times 10^8$ m/s and the free-space loss $L_p = 20\log_{10}(4\pi d/\lambda)$ dB. Give powers in dBm and decibel values to two decimals.

a) Find the occupied band, the noise power in that band and the receiver sensitivity $P_{\min}$.

b) Find the path loss at d_0, the received power and the link margin.

c) The carrier moves to 6 GHz and the antenna gains stay at 10 dBi. Find the range that keeps the margin of part (b). Then find that range if instead the antennas keep their size, so that each gain grows in proportion to f_c^2.

d) Back at 1.5 GHz, the bit rate rises to 32 Mb/s with the same scheme and roll-off. Find the new band, the new sensitivity and the range that keeps the same margin.

D5–16 · BIT ERROR AND GRAY LABELS

Three sets of eight equally likely points on the signal plane (ψ_1, ψ_2) are compared. Each point carries 3 bits, and the channel is AWGN with $\mathcal{N}(0, N_0/2)$. Set A is a square ring: $(\pm c, \pm c)$, $(\pm c, 0)$ and $(0, \pm c)$. Set B is three rows of a triangular grid: $(\pm b, \pm h)$, $(0, \pm h)$ and $(\pm \frac{b}{2}, 0)$, with $h = \frac{\sqrt{3}}{2} b$. Set C is eight levels on one axis: $\psi_1 = (2k - 9)\, g/2$ for $k = 1, \ldots, 8$. Set B comes with labels, read from left to right. The top row carries $\texttt{000}, \texttt{001}, \texttt{011}$, the middle row $\texttt{100}, \texttt{101}$ and the bottom row $\texttt{010}, \texttt{110}, \texttt{111}$.

a) Scale each set so that $E_b = 3$. Find c, b and g, and the $d_{\min}$ of each set.

b) A Gray labelling gives every pair at $d_{\min}$ labels that differ in one bit. For each set, give a Gray labelling or show that none exists.

c) Write the nearest-neighbour approximation of the bit error probability P_b of each set as a function of E_b/N_0. Use your labels from part (b), or the given labels for a set without a Gray labelling. Evaluate each at $E_b/N_0 = 9$.

d) Rank the three sets by P_b at high signal-to-noise ratio. Give the gap between neighbours in the ranking in decibels.

D5–17 · A CARRIER PHASE ERROR

A BPSK link sends $s_{1,2}(t) = \pm\sqrt{2E_b/T}\cos(2\pi f_c t)$ on $0 \le t \le T$ over an AWGN channel with $\mathcal{N}(0, N_0/2)$. The carrier reaches the receiver with an extra phase φ that the receiver does not know. The receiver correlates with $\psi_1(t) = \sqrt{2/T}\cos(2\pi f_c t)$ and $\psi_2(t) = -\sqrt{2/T}\sin(2\pi f_c t)$. It decides from the sign of the ψ_1 output alone. Take $E_b/N_0 = 8$.

a) For $\varphi = 25^\circ$, find the noiseless outputs on ψ_1 and ψ_2 in units of $\sqrt{E_b}$. Sketch the two points against the decision boundary.

b) Find the bit error probability as a function of φ and E_b/N_0 for $0 \le \varphi < 90^\circ$. Evaluate it at $\varphi = 0$, 25° and 60°.

c) Find the loss in decibels at $\varphi = 25^\circ$ and at $\varphi = 60^\circ$.

d) The phase error grows to $\varphi = 155^\circ$. Find the bit error probability, and say how differential encoding removes the problem.

D5-18 · SETS BUILT FROM TWO FREQUENCIES

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_{1,2}(t) = \pm 2\cos(2000\pi t), \qquad s_{3,4}(t) = \pm 2\cos(3000\pi t), \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Show that the two carriers are orthogonal on $0 \le t \le 1$, and choose an orthonormal basis.

b) Draw the signal constellation and the optimal decision regions for this signal constellation.

c) Determine the nearest-neighbour approximation as a function of $E_{s,\text{avg}}/N_0$. Evaluate it at 9 and compare with a four-point PSK set of the same energy.

D5-19 · SETS BUILT FROM TWO FREQUENCIES

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_1(t) = 0, \qquad s_2(t) = 3\cos(4000\pi t), \qquad s_3(t) = 3\cos(5000\pi t), \qquad 0 \le t \le 2.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Evaluate the approximation of part (b) at $E_{s,\text{avg}}/N_0 = 12$.

D5-20 · SENSITIVITY OF SEVERAL SCHEMES

A receiver is given a carrier channel of bandwidth $B = 6$ MHz. The pulses are raised cosines with roll-off $\alpha = 0.2$, so $B = (1+\alpha)R_s$, where R_s is the symbol rate. The receiver noise figure is $\mathrm{NF} = 6$ dB, and thermal noise at room temperature is $kT_0 = -174$ dBm/Hz. The target bit error probability is $P_b = 10^{-6}$. Three schemes with Gray labels are compared: QPSK, 8-PSK and 16-QAM. Use the nearest-neighbour form

$$P_b \approx \frac{\bar{N}_{\min}}{\log_2 M} Q\left(\sqrt{\frac{d_{\min}^2}{2N_0}}\right)$$

and the values $Q(4.753) = 1.00 \times 10^{-6}$, $Q(4.695) = 1.33 \times 10^{-6}$ and $Q(4.671) = 1.50 \times 10^{-6}$.

a) Find the symbol rate and the bit rate of each scheme.

b) Write $d_{\min}^2$ of each scheme in terms of E_b, and its P_b as a function of E_b/N_0. Find the E_b/N_0 in dB that each needs for $P_b = 10^{-6}$.

c) Find the sensitivity of each scheme in dBm.

d) Find the noise power in the 6 MHz band. Find the signal-to-noise ratio $P_{\min}/N$ at the QPSK sensitivity, and explain why it differs from the E_b/N_0 of part (b).

D5-21 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = 2\cos\left(2000\pi t + \frac{(k-1)\pi}{2}\right), \quad k \in \{1,2,3\},$$
$$s_4(t) = 4\cos\left(2000\pi t + \frac{3\pi}{2}\right),$$

all on $0 \le t \le 1$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Compare with a regular four-point PSK set of the same $E_{s,\text{avg}}$, in decibels.

D5-22 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_{1,2}(t) = \pm\sqrt{2}\cos(3000\pi t), \qquad s_{3,4}(t) = 2\sqrt{2}\cos\left(3000\pi t \pm \frac{\pi}{2}\right), \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

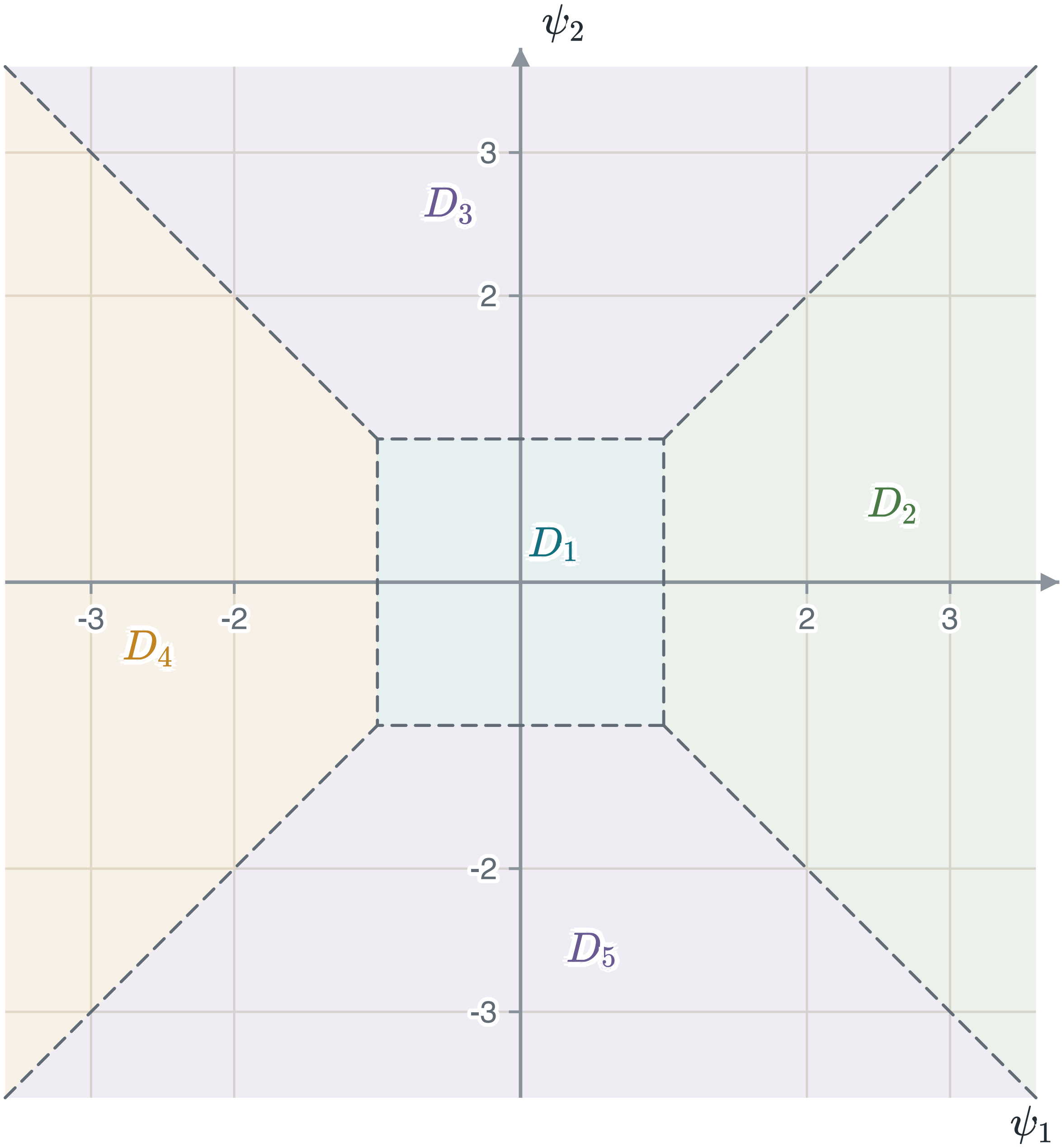
a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) The next distance is only slightly larger than $d_{\min}$. Add its terms to the approximation and evaluate both at $E_{s,\text{avg}}/N_0 = 11.25$.

D5-23 · REVERSED QUESTIONS AND COMPARISONS

Five equally probable waveforms are transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. The figure shows their optimal decision regions $D_1, \ldots, D_5$ on the basis $\psi_1(t) = \sqrt{2}\cos(2000\pi t)$, $\psi_2(t) = -\sqrt{2}\sin(2000\pi t)$, $0 \le t \le 1$. The central region is the square $|\psi_1| < 1$, $|\psi_2| < 1$, and the diagonals split the rest. One waveform is the zero signal. The other four have equal energy, and $E_{s,\text{avg}} = 3.2$.



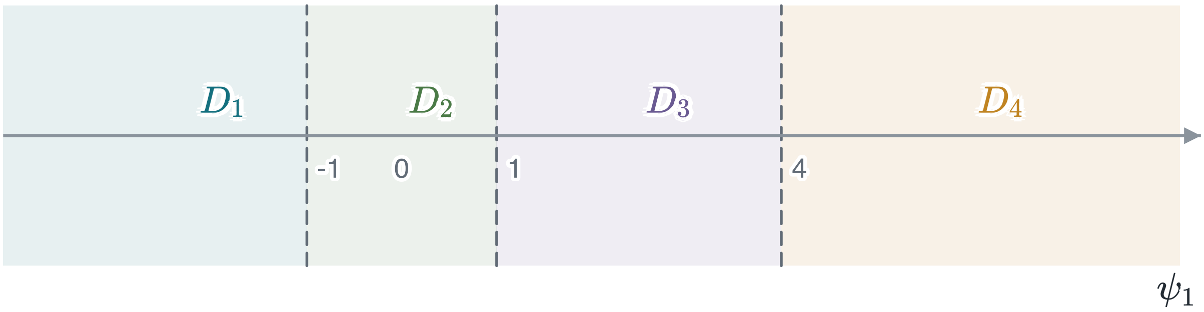
a) Locate the five signal points from the regions, and confirm the value of $E_{s,\text{avg}}$.

b) Write the five waveforms $s_1(t), \ldots, s_5(t)$, with s_k the symbol decided in D_k.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$. Evaluate it at $E_{s,\text{avg}}/N_0 = 14.4$.

D5-24 · REVERSED QUESTIONS AND COMPARISONS

Four equally probable waveforms $s_k(t) = c_k\sqrt{2}\cos(2000\pi t)$, $0 \le t \le 1$, with $c_1 < c_2 < c_3 < c_4$, are detected optimally after a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. The figure shows the decision thresholds on the axis $\psi_1(t) = \sqrt{2}\cos(2000\pi t)$. The smallest coefficient is $c_1 = -2$.



a) Find c_2, c_3 and c_4 from the thresholds.

b) Find $E_{s,\text{avg}}$ and write the four waveforms.

c) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$.

d) Give the four-point set with the same $d_{\min}$ and the least $E_{s,\text{avg}}$. Find its saving in decibels.

D5-25 · AMPLITUDE SETS ON ONE CARRIER

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_k(t) = c_k\sqrt{2}\cos(2000\pi t), \quad c_k \in \{-2,\ 0,\ 1,\ 3\}, \quad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\text{avg}}/N_0$, where $E_{s,\text{avg}}$ is the average symbol energy.

c) Evaluate the approximation of part (b) at $E_{s,\text{avg}}/N_0 = 28$.

d) Find the exact symbol error probability at the same value from the thresholds, and compare.

D5-26 · REVERSED QUESTIONS AND COMPARISONS

Two sets of sixteen equally probable waveforms are proposed, both on $0 \le t \le 1$:

$$\text{A: } s_k(t) = 2\sqrt{5}\cos\left(2000\pi t + \frac{k\pi}{8}\right),\ k \in \{1, \ldots, 16\},$$

$$\text{B: } s_{mn}(t) = \sqrt{2}\left[(2m-5)\cos(2000\pi t) - (2n-5)\sin(2000\pi t)\right],\ m, n \in \{1, 2, 3, 4\}.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Show that the two sets have the same $E_{s,\text{avg}}$, and draw both constellations with their optimal decision regions.

b) Determine the nearest-neighbour approximation of the average symbol error probability of each set as a function of $E_{s,\text{avg}}/N_0$.

c) Which set is better at equal $E_{s,\text{avg}}$, and by how many decibels? Evaluate both approximations at $E_{s,\text{avg}}/N_0 = 45$.

D5-27 · SETS BUILT FROM TWO FREQUENCIES

Consider a binary modulation scheme where the equally probable symbols have the following waveforms:

$$s_1(t) = 2\cos(2000\pi t), \qquad s_2(t) = 2\cos\left(2\pi(1000.75)t\right), \qquad 0 \le t \le 1.$$

These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Find the energies and the correlation coefficient $\rho = \langle s_1, s_2\rangle/E$. Neglect terms at the double frequency.

b) Use Gram–Schmidt to find the signal points, and draw the constellation with its optimal decision boundary.

c) Find the error probability as a function of $E_{s,\text{avg}}/N_0$. Compare it with the orthogonal choice $f_2 = 1000.5$ Hz at $E_{s,\text{avg}}/N_0 = 7.5$.

D5-28 · REVERSED QUESTIONS AND COMPARISONS

Eight equally probable waveforms are formed from two rings on $0 \le t \le 1$:

$$s_k(t) = \begin{cases} \sqrt{2}\cos\big(2000\pi t + (k-1)\frac{\pi}{2}\big), & k \in \{1,2,3,4\}, \\ R\sqrt{2}\cos\big(2000\pi t + (k-5)\frac{\pi}{2}\big), & k \in \{5,6,7,8\}, \end{cases}$$

with $R > 1$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Find $d_{\min}$ as a function of R.

b) Find the R that maximises $d_{\min}^2/E_{s,\text{avg}}$, and that maximum.

c) For this R, draw the constellation and regions, and determine the nearest-neighbour approximation as a function of $E_{s,\text{avg}}/N_0$.

d) Compare the result with 8-PSK.

D5-29 · REVERSED QUESTIONS AND COMPARISONS

A set of M equally probable waveforms $s(t) = a\sqrt{2}\cos(2000\pi t) - b\sqrt{2}\sin(2000\pi t)$, $0 \le t \le 1$, is transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$. The points (a, b) form a square grid centred on the origin. The nearest-neighbour approximation of the set is

$$P_e \approx 3\,Q\left(\sqrt{\frac{E_{s,\text{avg}}}{5N_0}}\right),$$

and $E_{s,\text{avg}} = 2.5$.

a) Find M and $d_{\min}$.

b) Give the values that a and b take, and confirm $N_{\min} = 3$.

c) Draw the signal constellation and the optimal decision regions for this signal constellation.

d) Find the bits per symbol and E_b, and write the approximation as a function of E_b/N_0.

D5-30 · AMPLITUDE-AND-PHASE SETS

Consider an M-ary modulation scheme where the equally probable symbols have the following waveforms:

$$s_0(t) = 0,$$
$$s_k(t) = 3\cos\left(3000\pi t + \frac{k\pi}{4}\right), \quad k \in \{1, \ldots, 8\},$$

all on $0 \le t \le 2$. These signals are planned to be transmitted over a standard AWGN channel with $\mathcal{N}(0, N_0/2)$.

a) Draw the signal constellation and the optimal decision regions for this signal constellation.

b) Determine the nearest-neighbour approximation of the average symbol error probability as a function of $E_{s,\mathrm{avg}}/N_0$, where $E_{s,\mathrm{avg}}$ is the average symbol energy.

c) Compare this set with 8-PSK at the same $E_{s,\mathrm{avg}}$, both in decibels and in bits per symbol.

MODULE

6

An Introduction to Information Theory

30 questions on an introduction to information theory. Write your reasoning in the space under each one.

D6–01 · SOFT AND HARD DECISIONS

A binary code of rate $R = \frac{3}{5}$ is sent with Gray-coded QPSK over a passband channel of bandwidth W. The noise is white and Gaussian with two-sided power spectral density $N_0/2$. The symbols use ideal Nyquist pulses with no excess bandwidth, so the link sends W symbols per second. With Gray coding, each QPSK symbol carries two coded bits, and each coded bit is detected like one BPSK bit. Let E_b be the energy per information bit, E_c the energy per coded bit and E_s the energy per symbol. A hard decision on each coded bit turns the link into a binary symmetric channel (BSC) with crossover probability $p = Q\left(\sqrt{2E_c/N_0}\right)$. Here $Q(x)$ is the probability that a zero-mean, unit-variance Gaussian variable exceeds x. The binary entropy is $H_b(p) = -p\log_2 p - (1-p)\log_2(1-p)$. Use $Q(1.302) = 0.0965$, $Q(1.409) = 0.0794$ and $H_b(0.0965) = 0.4578$. This entropy reaches $H_b(p) = 0.4$ at $p = 0.0794$.

- **a)** Express E_c and E_s in terms of E_b, and E_s/N_0 in dB in terms of E_b/N_0 in dB. Find the spectral efficiency $r = R_b/W$ of the link.
- **b)** The receiver keeps its matched-filter outputs as they are, which is a soft decision. Find the Shannon limit at efficiency r, the least E_b/N_0 in dB for reliable transmission.
- **c)** The receiver makes hard decisions instead. Reliable transmission then needs the BSC capacity $1 - H_b(p)$ to be at least R. Find the least E_b/N_0 in dB with hard decisions, and the loss in dB against part (b).
- **d)** The link runs at $E_b/N_0 = 1.5$ dB. Find p and the capacity of the BSC. Decide whether reliable transmission is possible with soft decisions and with hard decisions, and give each margin or shortfall in dB.

D6-02 · A FUNCTION OF ONE UNIFORM SOURCE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between 0 and 9. Let $Y \triangleq X^2 \pmod{10}$ be another DMS which is a function of X. Here $a \bmod m$ denotes the remainder of a on division by m, taken in $\{0, 1, \ldots, m-1\}$. In words, Y is the last decimal digit of X^2.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-03 · A FUNCTION OF ONE UNIFORM SOURCE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between 1 and 12. Let $Y \triangleq \lfloor 12/X \rfloor$ be another DMS which is a function of X. Here $\lfloor u \rfloor$ is the largest integer not greater than u.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-04 · A FUNCTION OF ONE UNIFORM SOURCE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between 0 and 15. Each value of X is written as a four-bit binary word. Let Y be the number of ones in that word. Y is another DMS which is a function of X.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-05 · A FUNCTION OF ONE UNIFORM SOURCE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between -5 and 5. Let $Y \triangleq \lceil |X|/2 \rceil$ be another DMS which is a function of X. Here $\lceil u \rceil$ is the smallest integer not less than u.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-06 · REPETITION CODING OVER A BINARY SYMMETRIC CHANNEL

A binary symmetric channel (BSC) flips each transmitted bit with probability $p = 0.05$, independently of the other bits. A message of $k = 20$ information bits is sent over it. A block error occurs when at least one of the 20 decoded bits is wrong. A repetition code of length n sends each information bit n times. The receiver then decides each bit by a majority vote over its n received copies. Use $H_b(0.05) = 0.2864$ bits, where $H_b(p) = -p \log_2 p - (1-p)\log_2(1-p)$.

a) The 20 bits are sent without coding. Calculate the probability of a block error.

b) Each bit is sent with a repetition code of length $n = 3$, and then of length $n = 5$. For each n, calculate the bit error probability after the majority vote and the block error probability.

c) Give the code rate of each repetition code, and the number of channel bits it uses for the 20 information bits.

d) Calculate the capacity of the channel. Compare the rates of part (c) with it, and state what the channel coding theorem allows.

D6-07 · THE SHANNON LIMIT OF THE BANDLIMITED CHANNEL

A company claims a modem that sends $R_b = 60$ Mb/s in a bandwidth of $W = 10$ MHz. The received signal power is $P = 1.8 \times 10^{-11}$ W. The noise is white and Gaussian with two-sided power spectral density $N_0/2$, where $N_0 = 4.0 \times 10^{-20}$ W/Hz. Treat the link as an ideal bandlimited channel with additive white Gaussian noise and no excess bandwidth.

a) Find the spectral efficiency $r = R_b/W$ of the claimed link. Then find the Shannon limit at that efficiency, the least E_b/N_0 in dB at which any system with it works reliably.

b) Calculate the energy per bit at the receiver and the actual E_b/N_0 in dB.

c) Decide whether the claim can be true, and by how many dB it misses or clears the limit. Confirm the verdict with the capacity $C = W \log_2(1 + P/(N_0 W))$.

d) The company then says the modem uses two separate channels at once, for example two pairs of antennas. Each has bandwidth 10 MHz and the same N_0, and the power and the bits are split equally. Repeat the test for one channel and give the new verdict.

D6-08 · A FUNCTION OF ONE UNIFORM SOURCE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between -7 and 7. Let $Y \triangleq \max(X, 0)$ be another DMS which is a function of X.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-09 · A FUNCTION OF TWO SOURCES, ONE WITH AN UNKNOWN CONSTANT

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable on the alphabet $\{1, 2, 3\}$. Let Z be a random variable (independent of X) with the probability mass function $p_Z(z) = c\left(\frac{1}{3}\right)^z$ for the integers $-1 \leq z \leq 1$. It is zero otherwise, and c is a constant. Finally, let $Y \triangleq X \times Z$ be another DMS which is a function of both X and Z.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-10 · A $(7,4)$ HAMMING CODE

A $(7,4)$ Hamming code places four data bits $d_1 d_2 d_3 d_4$ at positions 1 to 4 of a codeword $c_1 c_2 \cdots c_7$. Its three parity bits sit at positions 5 to 7. The code is defined by the parity-check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Row j of H is parity check j. It covers the positions where the row holds a 1, and it passes when the bits there hold an even number of ones. A codeword passes all three checks. For a received word, the syndrome $s_1 s_2 s_3$ has $s_j = 1$ when check j fails. The code is used on a binary symmetric channel (BSC) with crossover probability $p = 0.02$.

a) Write each parity bit in terms of the data bits. Find the codewords for the data `1010` and `0111`. Decide whether `1100011` and `0101100` are codewords.

b) Find the minimum distance $d_{\min}$ of the code and the number t of errors it can correct.

c) Write the syndrome of no error and of a single error at each position. Use this table to decode the received word `1101110`: give its syndrome, the codeword and the data bits.

d) The decoder of part (c) flips the bit its table names. Find the probability that a word is decoded correctly, the probability that no table entry fits, and the probability of a decoding error.

D6-11 · A FUNCTION OF TWO SOURCES, ONE WITH AN UNKNOWN CONSTANT

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable on the alphabet $\{0, 1, 2, 3\}$. Let Z be a random variable (independent of X) with the probability mass function $p_Z(z) = c\left(\frac{1}{2}\right)^z$ for the integers $0 \leq z \leq 2$. It is zero otherwise, and c is a constant. Finally, let $Y \triangleq \max(X, Z)$ be another DMS which is a function of both X and Z.

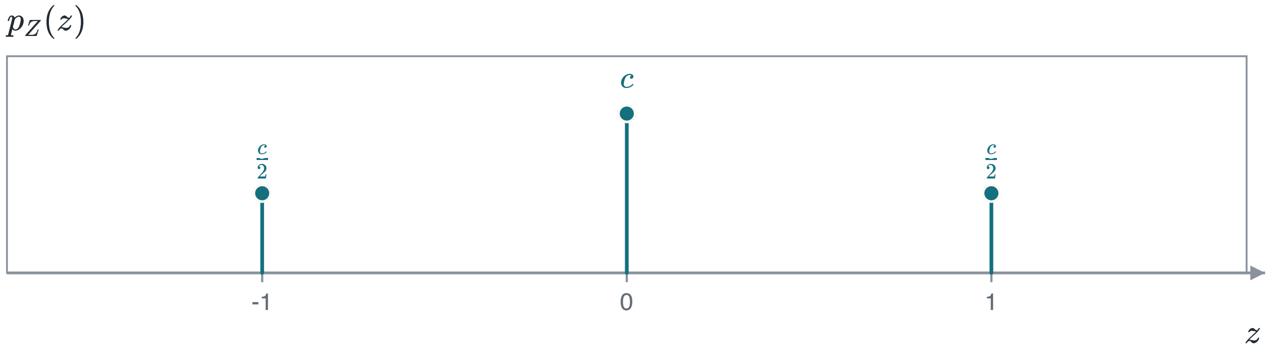
a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-12 · A FUNCTION OF TWO SOURCES, ONE WITH AN UNKNOWN CONSTANT

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable on the alphabet $\{0, 1, 2, 3\}$. Let Z be a random variable (independent of X) with the probability mass function $p_Z(z) = c\,2^{-|z|}$ for the integers $-1 \leq z \leq 1$. It is zero otherwise, and c is a constant. The pmf is shown below. Finally, let $Y \triangleq X + Z$ be another DMS which is a function of both X and Z.



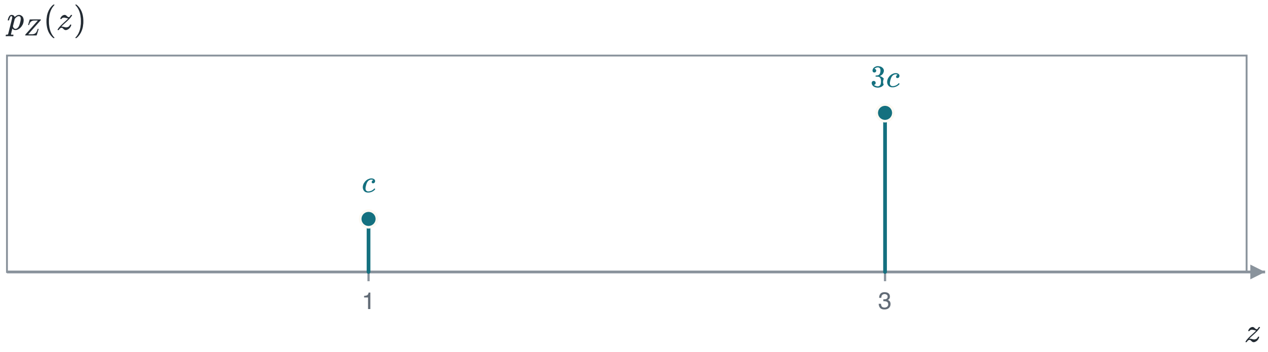
a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-13 · A FUNCTION OF TWO SOURCES, ONE WITH AN UNKNOWN CONSTANT

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable on the integers $0, 1, \ldots, 5$. Let Z be a random variable (independent of X) with the probability mass function $p_Z(z) = c\,z$ for $z \in \{1, 3\}$. It is zero otherwise, and c is a constant. The pmf is shown below. Finally, let $Y \triangleq \lfloor X/Z \rfloor$ be another DMS which is a function of both X and Z. Here $\lfloor u \rfloor$ is the largest integer not greater than u.



a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-14 · THE SHANNON LIMIT OF THE BANDLIMITED CHANNEL

Three uncoded constellations with Gray coding are used on a channel with additive white Gaussian noise: QPSK, 8-PSK and 16-QAM. Each must reach the bit error probability $P_b = 10^{-4}$. With ideal Nyquist pulses and no excess bandwidth, an M-point constellation sends $\log_2 M$ bits per second per hertz. The nearest-neighbour approximations of the bit error probability are $P_b = Q\big(\sqrt{2E_b/N_0}\big)$ for QPSK, $P_b \approx \frac{2}{3}\,Q\big(\sqrt{0.879\,E_b/N_0}\big)$ for 8-PSK and $P_b \approx \frac{3}{4}\,Q\big(\sqrt{0.8\,E_b/N_0}\big)$ for 16-QAM. Here $Q(x)$ is the probability that a zero-mean, unit-variance Gaussian variable exceeds x. A table gives $Q(3.719) = 1.00 \times 10^{-4}$, $Q(3.646) = 1.33 \times 10^{-4}$ and $Q(3.615) = 1.50 \times 10^{-4}$.

a) For each constellation, find the E_b/N_0 in dB that gives $P_b = 10^{-4}$.

b) Give the spectral efficiency r of each constellation. Then find the Shannon limit at that efficiency, the least E_b/N_0 in dB at which any system with it works reliably.

c) How far is each constellation from its Shannon limit, in dB? Name the constellation farthest from its limit, and give that gap as a ratio of energies per bit.

D6-15 · A FUNCTION OF TWO SOURCES, ONE WITH AN UNKNOWN CONSTANT

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable on the alphabet $\{0, 1, 2, 3\}$. Let Z be a random variable (independent of X) with the probability mass function $p_Z(z) = c\left(\frac{1}{2}\right)^z$ for the integers $0 \leq z \leq 3$. It is zero otherwise, and c is a constant. Finally, let $Y \triangleq \min(X, Z)$ be another DMS which is a function of both X and Z.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-16 · TWO INDEPENDENT SOURCES DESCRIBED BY RATIOS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{0, 1, 2\}$, where $P(X = 0) = 2P(X = 1) = 4P(X = 2)$. Similarly, the source Y is described by the alphabet $\mathcal{Y} = \{0, 1, 2, 3\}$, where each symbol is generated with equal probability. Let $Z \triangleq X + Y$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for the source, Z.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-17 · TWO INDEPENDENT SOURCES DESCRIBED BY RATIOS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{1, 2, 3\}$, where each symbol is generated with equal probability. Similarly, the source Y is described by the alphabet $\mathcal{Y} = \{1, 2, 3\}$, where $P(Y = 1) = 2P(Y = 2) = 2P(Y = 3)$. Let $Z \triangleq X \times Y$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for the source, Z.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-18 · TWO INDEPENDENT SOURCES DESCRIBED BY RATIOS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{0, 1, 2, 3\}$, where each symbol is generated with equal probability. Similarly, the source Y is described by the alphabet $\mathcal{Y} = \{0, 1, \ldots, 5\}$, where each symbol is generated with equal probability. Let $Z \triangleq |X - Y|$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for the source, Z.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-19 · TWO INDEPENDENT SOURCES DESCRIBED BY RATIOS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{1, 2, 3, 4, 5\}$, where each symbol is generated with equal probability. Similarly, the source Y is described by the same alphabet, where each symbol is generated with equal probability. Let $Z \triangleq \min(X, Y)$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for the source, Z.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-20 · TWO INDEPENDENT SOURCES DESCRIBED BY RATIOS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{0, 1, 2\}$, where $P(X = 1) = 2P(X = 0) = 2P(X = 2)$. Similarly, the source Y is described by the alphabet $\mathcal{Y} = \{0, 1, 2\}$, where $P(Y = 0) = 2P(Y = 1) = 4P(Y = 2)$. Let $Z \triangleq X - Y$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for the source, Z.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-21 · TWO INDEPENDENT SOURCES DESCRIBED BY RATIOS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{1, 2, 3, 4\}$, where $P(X = k) = k\,P(X = 1)$. Similarly, the source Y is described by the alphabet $\mathcal{Y} = \{0, 2, 4\}$, where each symbol is generated with equal probability. Let $Z \triangleq X + Y$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for the source, Z.

c) Calculate the coding efficiency of the Huffman code designed in part (b).

D6-22 · THE SHANNON LIMIT OF THE BANDLIMITED CHANNEL

An adaptive link uses a binary code of rate $\frac{1}{2}$ with one of three constellations: QPSK, 16-QAM or 64-QAM. The passband channel has bandwidth $W = 6$ MHz. The symbols use ideal Nyquist pulses with no excess bandwidth, so the link sends W symbols per second. Each coded scheme works 1.5 dB above the Shannon limit for its spectral efficiency. The transmit power and the noise are fixed, and the received power falls as $1/d^2$ with the distance d.

a) Calculate the information bit rate R_b of each scheme and its spectral efficiency $r = R_b/W$.

b) Find the minimum E_s/N_0 in dB that each scheme needs, where E_s is the energy per symbol.

c) QPSK reaches the largest range, 2.4 km. Find the ranges of the other two schemes.

d) A user is at $d = 1.2$ km. Which scheme gives the highest bit rate there, and with how many dB to spare?

D6-23 · JUDGING A CODE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between 1 and 16. Let $Y \triangleq \lfloor \log_2 X \rfloor$ be another DMS which is a function of X. Here $\lfloor u \rfloor$ is the largest integer not greater than u.

a) Calculate the entropy of the source, Y.

b) Design a *binary* Huffman code for the source, Y.

c) A fixed-length binary code is proposed for Y instead. Give its codeword length and its coding efficiency.

d) Calculate the coding efficiency of the Huffman code of part (b), and explain the value.

D6-24 · JUDGING A CODE

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{0, 1, 2, 3\}$, where each symbol is generated with equal probability. Similarly, the source Y is described by the same alphabet, where each symbol is generated with equal probability. Let $Z \triangleq |X - Y|$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Design a *binary* Huffman code for Z, placing each merged probability as high as possible in the list.

c) Design a second binary Huffman code, placing each merged probability as low as possible. Show that both codes have the same average length.

d) Calculate the coding efficiency and the variance of the codeword lengths of both codes. Which code suits a transmitter with a finite buffer?

D6-25 · CODING THE SECOND-ORDER EXTENSION

Let X be a DMS which is modeled as a uniform random variable on $\{0, 1, 2, 3\}$. Let Z be independent of X and uniform on $\{0, 1\}$. Let $Y \triangleq (X \times Z) \pmod{2}$ be another DMS, the parity of the product.

a) Calculate the probabilities and the entropy of the source, Y.

b) Design a *binary* Huffman code for Y and calculate its coding efficiency.

c) Design a binary Huffman code for the second-order extension of Y, whose symbols are pairs Y_1Y_2 of successive outputs.

d) Calculate the average number of bits per symbol of Y and the coding efficiency for the code of part (c). Compare with part (b).

D6-26 · THE ENTROPY OF A FUNCTION AGAINST ITS INPUTS

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between -2 and 5. Let $Y \triangleq |X - 1|$ be another DMS which is a function of X.

a) Calculate the entropies $H(X)$ and $H(Y)$.

b) Calculate the mutual information $I(X;Y)$, the conditional entropy $H(X \mid Y)$ and the joint entropy $H(X,Y)$.

c) Design a *binary* Huffman code for the source, Y.

d) Calculate the coding efficiency of the Huffman code designed in part (c).

D6-27 · THE ENTROPY OF A FUNCTION AGAINST ITS INPUTS

Consider two *independent* discrete memoryless sources, X and Y. The source X is described by the alphabet $\mathcal{X} = \{0, 1, 2\}$, where $P(X = 0) = 2P(X = 1) = 2P(X = 2)$. Similarly, the source Y is described by the alphabet $\mathcal{Y} = \{0, 1, 2, 3\}$, where each symbol is generated with equal probability. Let $Z \triangleq X + Y$ be another discrete memoryless source.

a) Calculate the entropy of the source, Z.

b) Calculate the conditional entropy $H(Z \mid X)$ and the mutual information $I(X; Z)$.

c) Design a *binary* Huffman code for the source, Z.

d) Calculate the coding efficiency of the Huffman code designed in part (c).

D6-28 · JUDGING A CODE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between 1 and 9. Let $Y \triangleq X \pmod 4$ be another DMS which is a function of X. Here $a \bmod m$ denotes the remainder of a on division by m, taken in $\{0, 1, \ldots, m-1\}$. A designer proposes the code $Y = 1 \to$ 0, $Y = 0 \to$ 1, $Y = 2 \to$ 01, $Y = 3 \to$ 10.

a) Calculate the entropy of the source, Y.

b) Use the Kraft inequality to decide whether a prefix code with the lengths of the proposed code exists. Give a bit string that the proposed code cannot decode uniquely.

c) Design a *binary* Huffman code for the source, Y.

d) Calculate the coding efficiency of the Huffman code, and compare it with a fixed-length code for Y.

D6-29 · JUDGING A CODE

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable taking the integer values between 1 and 12. Let $Y \triangleq X \pmod 5$ be another DMS which is a function of X. Here $a \bmod m$ denotes the remainder of a on division by m, taken in $\{0, 1, \ldots, m-1\}$.

a) Calculate the entropy of the source, Y.

b) Give each symbol the length $l(y) = \left\lceil \log_2 \frac{1}{P(Y=y)} \right\rceil$. Check the Kraft inequality for these lengths and calculate their average length.

c) Design a *binary* Huffman code for the source, Y.

d) Calculate the coding efficiency of both codes. Which one is optimal?

D6-30 · A FUNCTION OF TWO SOURCES, ONE WITH AN UNKNOWN CONSTANT

Let X be a discrete memoryless source (DMS) which is modeled as a uniform random variable on the alphabet $\{0, 1, 2, 3\}$. Let Z be a random variable (independent of X) with the probability mass function $p_Z(z) = c\, a^z$ for $z \in \{0, 1, 2\}$. It is zero otherwise, and c and $0 < a < 1$ are constants. Finally, let $Y \triangleq X \times Z$ be another DMS which is a function of both X and Z. It is known that $P(Y = 0) = \frac{10}{13}$.

a) Find the constants c and a.

b) Calculate the entropy of the source, Y.

c) Design a *binary* Huffman code for the source, Y.

d) Calculate the coding efficiency of the Huffman code designed in part (c).

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